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# Randomized collective choices based on a fractional tournament

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An *extension rule* assigns to each fractional tournament $x$ (specifying, for every pair of social alternatives $a$ and $b$, the proportion $x_{ab}$ of voters who prefer $a$ to $b$) a random choice function $y$ (specifying a collective choice probability distribution for each subset of alternatives), which chooses $a$ from $\{a, b\}$ with probability $x_{ab}$.

There exist multiple neutral and stochastically rationalizable extension rules. Both linearity (requiring that $y$ be an affine function of $x$) and independence of irrelevant comparisons (asking that the probability distribution on a subset of alternatives depend only on the restriction of the fractional tournament to that subset) are incompatible with very weak properties implied by stochastic rationalizability.

We identify a class of maximal domains, which we call *sequentially binary*, guaranteeing that every fractional tournament arising from a population of voters with preferences in such a domain has a unique admissible stochastically rationalizable extension.

Keywords. Voting, fractional tournament, random choice, stochastic rationalizability.

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## 1. Introduction

Randomizing collective decisions helps reconcile fairness and rationality. The random dictatorship mechanism treats all participating individuals equally and produces collective choices that are rationalizable in the sense of the random utility model pioneered by Block and Marschak (1960). Moreover, as Barberà and Sonnenschein (1978) and McLennan (1980) point out, the probability of selecting an alternative $a$ from a pair $\{a, b\}$ only depends on the restriction of the preference profile to that pair of alternatives.[1]

Applying the random dictatorship mechanism, however, requires to know the distribution of preferences in the population. In practice, voters are often asked to cast secret ballots on binary choice problems. In such cases, the information available to the collective decision maker takes the form of a *fractional tournament* giving, for any alternatives $a$ and $b$, the proportion $x_{ab}$ of voters who support $a$ against $b$. This is more informative than a majority tournament (recording only whether or not a majority supports

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[1]This natural stochastic generalization of Arrow's IIA axiom and the obvious unanimity condition essentially characterize the random dictatorship rules; see McLennan (1980) for details.

$a$ against $b$) but less informative than a preference distribution. Based on the fractional tournament $x$, what randomized choice behavior should the collective decision maker adopt?[2]

If the choice is between $a$ and $b$, she should arguably respect $x$ and pick $a$ with probability $x_{ab}$.[3] But what are the appropriate randomized choices from larger agendas? Answering that question amounts to constructing an *extension rule* that transforms $x$ into a collective random choice function $y$. The subject of this paper is the axiomatic analysis of such rules.

The central axiom we are interested in stipulates that $y$ should be *(stochastically) rationalizable*. In contrast to the deterministic setup where the majority tournament arising from a population of rational voters cannot generally be extended to a deterministic rational choice function, stochastic rationalizability is feasible. But because the distribution of preferences generating $x$ cannot generally be recovered uniquely, $x$ admits several stochastically rationalizable extensions; see Section 3. This multiplicity problem can be handled in two ways. One consists in imposing axioms that complement stochastic rationalizability, the other is to identify domain conditions under which $x$ does possess a unique stochastically rationalizable extension.

Section 4 follows the first route. We start with the basic axiom of neutrality and show that there exist multiple neutral and stochastically rationalizable extension rules. See Theorem 1 and the Appendix.

Several other natural axioms are incompatible with very weak properties implied by stochastic rationalizability. Linearity (requiring that $y$ be an affine function of $x$) conflicts with the requirement that $y$ be rationalizable when $x$ is generated by a population of unanimous voters. This incompatibility persists if linearity is replaced with the related property of betweenness preservation (asking that the extension of a fractional tournament $x''$ that is between $x$ and $x'$ should be between the extensions of $x$ and $x'$); see Theorem 2.

Next, the important axiom of independence of irrelevant comparisons (asking that the probability distribution recommended by $y$ on a subset of alternatives be determined by the restriction of $x$ to that subset) is incompatible with agenda monotonicity (requiring that the probability of selecting an alternative does not increase when the agenda expands); see Theorem 3.

Section 5 follows the second route. We study a class of preference domains that we call *sequentially binary*. Each ordering in such a domain is constructed from bottom to top through a sequence of $m-1$ binary choices, where $m$ denotes the number of alternatives. The first choice determines which of two exogenously specified alternatives is the worst alternative in the ordering. The second binary choice (which depends upon the outcome of the first) determines the second-worst alternative, and so on. There are restrictions tying the successive choices, which will be explained in Section 5.

[2] In practice, the number $x_{ab}$ may be known only for some pairs $(a, b)$. Dealing with incomplete fractional tournaments is beyond the scope of this paper.

[3] Note that this is indeed the outcome of the random dictatorship mechanism for *any* distribution of preferences generating $x$.

Although a fractional tournament $x$ may be generated by several probability distributions on a given sequentially binary domain, it turns out that all such distributions generate the same random choice function. This implies that there is a unique admissible extension rule on the set of fractional tournaments generated by preferences in a sequentially binary domain; see Theorem 4 and its corollary. Theorem 5 states that this rule satisfies linearity and independence of irrelevant comparisons.

## 2. Extension rules

Given a finite set $A \subseteq \mathbb{N} = \{1, 2, \dots\}$ containing $m \geq 2$ *alternatives*, let $\mathcal{B}_A = \{(a, b) \in A \times A | a \neq b\}$, and let

$$X = \left\{x \in [0, 1]^{\mathcal{B}_A} | x(a, b) + x(b, a) = 1 \text{ for all } (a, b) \in \mathcal{B}_A\right\}.$$

For every $(a, b) \in \mathcal{B}_A$, we write $x(a, b)$ as $x_{ab}$ and interpret this number as the proportion of voters who prefer alternative $a$ to $b$. A point $x = (x_{ab})_{(a,b)\in\mathcal{B}_A} \in X$ is a *fractional tournament* (on $A$).

We restrict our attention to the fractional tournaments that are generated by a population of rational voters. Formally, let $\mathcal{P}$ denote the set of (linear) preference orderings[4] on $A$ and let $\Delta(\mathcal{P}) = \{\alpha \in [0, 1]^{\mathcal{P}} | \sum_{P\in\mathcal{P}} \alpha(P) = 1\}$ denote the set of probability distributions on $\mathcal{P}$. The fractional tournament $x^*(\alpha)$ generated by $\alpha \in \Delta(\mathcal{P})$ is defined by

$$x^*_{ab}(\alpha) = \sum_{P\in\mathcal{P}:aPb} \alpha(P) \tag{1}$$

for all $(a, b) \in \mathcal{B}_A$. We call a fractional tournament $x \in X$ *rationalizable* if $x = x^*(\alpha)$ for some $\alpha \in \Delta(\mathcal{P})$. We denote by $X^*$ the set of rationalizable fractional tournaments (on $A$). Identifying the distribution putting probability one on a single ordering $P$ with $P$ itself, $x^*(P)$ denotes the (degenerate) rationalizable fractional tournament

$$x^*_{ab}(P) = \begin{cases} 1 & \text{if } aPb, \\ 0 & \text{otherwise.} \end{cases}$$

Since $x^*(P)$ is generated by a population of voters having the same preference, we call it a *unanimous tournament*. By definition,

$$x^*(\alpha) = \sum_{P\in\mathcal{P}} \alpha(P) x^*(P).$$

Thus, $X^* = co\{x^*(P) | P \in \mathcal{P}\}$: the rationalizable fractional tournaments form the convex hull of the unanimous tournaments.

[4]A linear ordering on $A$ is a binary relation $P \subseteq A \times A$ that is *complete* (for all distinct $a, b \in A$, $(a, b) \in P$ or $(b, a) \in P$), *asymmetric* (for all $a, b \in A$, $(a, b) \in P \Rightarrow (b, a) \notin P$), and *transitive* (for all $a, b, c \in A$, $[(a, b) \in P \text{ and } (b, c) \in P] \Rightarrow (a, c) \in P$). Our use of the term "linear ordering" is slightly nonstandard. In particular, $P$ is *irreflexive* (for all $a \in A$, $(a, a) \notin P$).

For $m = 3$, $X^*$ consists of all $x \in X$ such that $1 \leq x_{12} + x_{23} + x_{31} \leq 2$. In general, $X^*$ is a convex polytope of dimension $m(m-1)/2$.[5]

Let $\mathcal{S}_A = \{B \subseteq A \mid |B| \geq 2\}$. For each *agenda* $B \in \mathcal{S}_A$, let $\Delta(B) = \{y_B \in [0,1]^B \mid \sum_{a \in B} y_{aB} = 1\}$ be the set of probability distributions on $B$, and let

$$Y = \prod_{B \in \mathcal{S}_A} \Delta(B).$$

A point $y = (y_B)_{B \in \mathcal{S}_A} \in Y$ is a *random choice function* (on $A$). For each $B \in \mathcal{S}_A$ and $a \in B$, the number $y_{aB}$ is the probability with which society chooses $a$ when the set of feasible alternatives is $B$.

A random choice function $y \in Y$ *extends* (or is an extension of) a rationalizable fractional tournament $x \in X^*$ if $y_{a\{a,b\}} = x_{ab}$ for all $(a, b) \in \mathcal{B}_A$. An *extension rule* is a function $f : X^* \to Y$ such that $f(x)$ extends $x$ for every $x \in X^*$.

An extension rule is a mathematical object that formally describes a particular type of collective decision mechanism. We submit that (i) the input of a collective decision mechanism is often adequately described by a rationalizable fractional tournament $x$, and (ii) a random choice function $y$ extending $x$ may be a sensible output of a collective decision mechanism.

(i) From a positive viewpoint, modeling the input of a collective choice mechanism as a fractional tournament makes sense because real-life procedures often request voters to *secretly* express their opinion on *binary* agendas. The restriction to binary agendas may stem from the concern that voters might find it difficult to elicit their preference ranking over larger agendas, and from the resulting willingness to submit a simple question to the popular vote. Secrecy may reflect an effort to avoid attempts by some voters to influence the vote of others. Whatever the reasons for using a secret binary protocol are, the information revealed by such a protocol is very close to a fractional tournament. The collective decision maker knows, for each pair of alternatives $\{a, b\}$ submitted to the voters, the number of those who prefer $a$ to $b$, the number of those who prefer $b$ to $a$, and the number of those who did not express a valid preference.

Admittedly, this is not quite a fractional tournament. On the one hand, by restricting attention to the proportion of valid ballots in favor of each alternative in the pair $\{a, b\}$, a fractional tournament ignores the (arguably irrelevant) size of the electorate and the (possibly relevant) fraction of voters who did not submit a valid ballot. On the other hand, a fractional tournament assumes that voters are consulted on *all* pairs of alternatives, which is not the case in practice. We view the analysis of extension rules as a step in the analysis of secret binary protocols.

There is a sizable literature on what Fishburn (1977) dubs *C2* (or *pairwise*) *social choice functions*. These are decision mechanisms, which require more information than the majority relation generated by the voters' preference profile, yet only use the matrix $p = (p_{ab})_{(a,b) \in \mathcal{B}_A}$ generated by that profile, where $p_{ab}$ is the number of voters who

[5]For any $m$, it is easy to see that $x \in X^*$ only if $1 \leq x_{ab} + x_{bc} + x_{ca} \leq 2$ for all distinct $a, b, c \in A$. Dridi (1980) proved that these *triangle inequalities* imply $x \in X^*$ when $m \leq 5$, but not when $m > 5$. Identifying a minimal set of linear inequalities guaranteeing that a fractional tournament is rationalizable remains an open problem for $m > 6$; see Fishburn (1992) and Martí and Reinelt (2011).

prefer $a$ to $b$. Compared to $p$, the fractional tournament $x$ is a slightly less informative summary of the voters' preference profile that ignores the total number of voters.[6]

The foregoing discussion vindicates our interest in fractional tournaments. The *rationalizable* ones arise if voters' preferences are assumed to be linear orderings over the set of alternatives. This rationality assumption is central in deterministic social choice theory. Indeed, a fundamental insight of the theory is that a deterministic mechanism taking into account the diverse preferences of rational individuals cannot produce rationalizable collective decisions. We maintain the individual rationality assumption because we wish to examine to what extent the tension between representativeness and collective rationality persists when collective choices can be randomized. As we shall see, *stochastic* collective rationalizability is achievable.

(ii) This brings us to the issue of modeling the output of collective decision mechanisms. As suggested in the previous paragraph, our motivation for studying random choice functions that extend a rationalizable fractional tournament is normative.

It is generally impossible to derive a satisfactory deterministic choice function from a rationalizable fractional tournament because of the possibility of Condorcet cycles: if two-thirds of the voters prefer alternative 1 to alternative 2, two-thirds prefer 2 to 3, and two-thirds prefer 3 to 1, there is no neutral and anonymous way of selecting a single alternative from the set $\{1, 2, 3\}$. Randomized collective choices are therefore necessary to ensure impartiality; see Fishburn (1984) and Brandl, Brandt, and Seedig (2016) for a more detailed discussion.

The study of randomized collective choice mechanisms is not new. Zeckhauser (1969) was the first to formalize them as mappings from preference profiles to lotteries over the (fixed) set of alternatives. Some of these mechanisms only use the information contained in the associated matrix of majority margins: an important example are the mechanisms selecting a *maximal lottery*, namely one that is weakly preferred to every other by an expected majority of voters; see Kreweras (1965), Fishburn (1984), and Brandl, Brandt, and Seedig (2016).

Randomization, however, is generally regarded as a necessary evil: its only purpose is the impartial resolution of ties. The current paper takes the view that lotteries are also useful as a tool to reflect the diversity of voters' opinions and avoid the "tyranny of the majority." This motivates our search for an *extension* of the fractional tournament generated by the voters' preferences: even if a majority prefers alternative 1 to alternative 2, it may make sense to choose 2 with a probability equal to the proportion of voters who prefer 2 to 1.

Extension rules have not received much attention in the literature. The only example we are aware of is the *proportional Borda rule* mentioned by Brandt (2017), which selects

[6]The matrix of majority margins $q = (q_{ab})_{(a,b)\in\mathcal{B}_A} := (p_{ab} - p_{ba})_{(a,b)\in\mathcal{B}_A}$ is a *weighted tournament*, i.e., a point $q \in \mathbb{N}^{\mathcal{B}_A}$ such that $q_{ab} + q_{ba} = 0$ for all distinct $a, b \in A$. Debord (1987) shows that if all components $q_{ab}$ of a weighted tournament $q$ have the same parity, there is a profile of linear preference orderings whose matrix of majority margins coincides with $q$. Recent work on weighted tournaments and pairwise social choice functions includes De Donder, Le Breton, and Truchon (2000) and Fischer, Hudry, and Niedermeier (2016).

alternative $a$ from agenda $B$ with a probability equal to $a$'s Borda score relative to $B$. It can be computed from the fractional tournament $x$ generated by the voters' preferences:

$$f^{\text{Borda}}_{aB}(x) = \frac{\sum_{c \in B \setminus a} x_{ac}}{\sum_{b \in B} \sum_{c \in B \setminus b} x_{bc}} = \frac{\sum_{c \in B \setminus a} x_{ac}}{\binom{|B|}{2}}$$

for all $B \in \mathcal{S}_A$ and $a \in B$.[7] As we shall see in the next section, this random choice function is not stochastically rationalizable in the sense usually given to that term.

## 3. Stochastic rationalizability

Over 60 years ago, Block and Marschak (1960) formulated a stochastic generalization of the notion of rationality known in mathematical psychology as the random utility model. The model spurred enormous interest and constitutes today the conventional interpretation of stochastic rationalizability. Luce and Suppes (1965) is a classic introduction to the literature. A random choice function is (stochastically) rationalizable if it maximizes a randomly selected ordering: there exists a probability distribution $\alpha$ over the set of linear orderings on $A$ such that the probability of choosing alternative $a$ from an agenda $B$ coincides with the probability of drawing at random (according to the distribution $\alpha$) a linear ordering whose best alternative in $B$ is $a$.

Formally, the random choice function $y^*(\alpha) \in Y$ generated by $\alpha \in \Delta(\mathcal{P})$ is given by

$$y^*_{aB}(\alpha) = \sum_{P \in \mathcal{P}: aPb \text{ for all } b \in B \setminus \{a\}} \alpha(P) \tag{2}$$

for all $B \in \mathcal{S}_A$ and $a \in B$. A random choice function $y \in Y$ is *(stochastically) rationalizable* if $y = y^*(\alpha)$ for some $\alpha \in \Delta(\mathcal{P})$. We let $Y^*$ denote the set of (stochastically) rationalizable random choice functions. Identifying the distribution putting probability one on $P$ with $P$ itself, $y^*(P)$ denotes the random choice function which selects the best feasible alternative according to $P$ with probability one:

$$y^*_{aB}(P) = \begin{cases} 1 & \text{if } aPb \text{ for all } b \in B \setminus \{a\}, \\ 0 & \text{otherwise.} \end{cases}$$

Using this notation,

$$y^*(\alpha) = \sum_{P \in \mathcal{P}} \alpha(P) y^*(P).$$

The random choice function $y^*(\alpha)$ can be interpreted as a "random dictatorship" maximizing each preference $P$ with probability $\alpha(P)$.

[7] Brandt (2017) considers fixed-agenda mechanisms, i.e., functions mapping each fractional tournament to a probability distribution over the set $A$. The rule $f^{\text{Borda}}$ is a variable-agenda version of the mechanism he describes.

Throughout this paper, we focus on extension rules that produce rationalizable random choice functions.

*Stochastic rationalizability* $f(X^*) \subseteq Y^*$.

Recall that $X^*$ stands for the set of *rationalizable* fractional tournaments. The above axiom thus stipulates that deterministic rationality at the voters' level should translate into stochastic rationalizability at the collective level. Our insistence on collective rationalizability follows the Arrovian tradition. It is motivated by the view that, to ensure the continuing participation of rational voters, collective decisions should be reasonably consistent across agendas. For instance, the probability of choosing an alternative should not increase when the agenda expands. Stochastic rationalizability guarantees several such consistency properties; in fact, it is uniquely characterized by a collection of consistency properties identified by Falmagne (1978).

Stochastic rationalizability is not an innocuous requirement. The proportional Borda rule, for instance, violates it. To see why, suppose $A = \{1, 2, 3\}$ and write an ordering by listing (without commas) the alternatives from best to worst: for instance, 123 denotes the ordering $P = \{(1, 2), (2, 3), (1, 3)\}$. The only stochastically rationalizable extension of the unanimous tournament $x^*(123)$ is the degenerate random choice function $y^*(123)$. From the set $A$, this random choice function picks 1 with probability 1. By contrast, the Borda random choice function $f^{\text{Borda}}(x^*(123))$ picks 1 with probability 2/3 and 2 with probability 1/3.

Yet, stochastic rationalizability is feasible. Indeed, any probability distribution generating a given fractional tournament also generates a stochastically rationalizable extension of it: for any $x \in X^*$ and any $\alpha \in \Delta(\mathcal{P})$ such that $x = x^*(\alpha)$, the random choice function $y^*(\alpha)$ is a stochastically rationalizable extension of $x$.

The central difficulty is that different probability distributions generating the same fractional tournament may generate different random choice functions. As an illustration, suppose $A = \{1, 2, 3\}$ and recall from Footnote 5 that $X^*$ can be identified with the set of all $x = (x_{12}, x_{23}, x_{31}) \in [0, 1]^3$ such that $1 \le x_{12} + x_{23} + x_{31} \le 2$. Using this notation, the fractional tournament $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in X^*$ can be written as

$$x = \frac{1}{2}(1, 1, 0) + \frac{1}{2}(0, 0, 1) = \frac{1}{2}x^*(123) + \frac{1}{2}x^*(321) \tag{3}$$

or

$$x = \frac{1}{2}(1, 0, 0) + \frac{1}{2}(0, 1, 1) = \frac{1}{2}x^*(132) + \frac{1}{2}x^*(231). \tag{4}$$

Under the first decomposition, $x$ is generated by a population $\alpha$ in which one-half of the voters have preference 123 and the other half have the opposite preference 321. Under the second decomposition, $x$ is generated by a population $\alpha'$ in which one-half of the voters have preference 132 and the other half have preference 231. Although $x^*(\alpha) = x^*(\alpha') = x$, the random choice functions $y^*(\alpha), y^*(\alpha')$ generated by the two distributions $\alpha, \alpha'$ differ. The random choice function $y^*(\alpha) = \frac{1}{2}y^*(123) + \frac{1}{2}y^*(321)$ chooses alternatives 1 and 3 from the set $\{1, 2, 3\}$ with probability 1/2 each whereas $y^*(\alpha') = \frac{1}{2}y^*(132) + \frac{1}{2}y^*(231)$ picks alternatives 1 and 2 with probability 1/2 each.

The multiplicity of stochastically rationalizable extensions is the rule rather than an exception. When $m = 3$, $x \in X^*$ has several stochastically rationalizable extensions if and only if $x$ belongs to the interior of $X^*$. To see why, check first that any interior $x$ can be generated by different distributions: there exist $\alpha, \alpha' \in \Delta(\mathcal{P})$, $\alpha \neq \alpha'$, such that $x = x^*(\alpha) = x^*(\alpha')$. Since the random utility model is identified when there are only three alternatives (Block and Marschak (1960)), $y^*(\alpha) \neq y^*(\alpha')$. But both $y^*(\alpha)$ and $y^*(\alpha')$ extend $x$. Conversely, any $x$ not in the interior of $X^*$ is generated by a unique distribution and possesses a unique stochastically rationalizable extension.

## 4. Results

Stochastic rationalizability is a property of the random choice function $y$ associated with a given fractional tournament $x$. We now explore the possibility of combining stochastic rationalizability with restrictions on how $y$ changes with $x$.

### 4.1 *Neutral extension rules*

The axiom of *neutrality* requires that all alternatives be treated equally. Formally, let $\Pi$ denote the set of bijections from $A$ to itself. For every $\pi \in \Pi$, $x \in X^*$, and $y \in Y^*$, define $x^\pi \in X$ by

$$x^\pi_{\pi(a)\pi(b)} = x_{ab} \quad \text{for all } (a, b) \in \mathcal{B}_A,$$

and define $y^\pi \in Y$ by

$$\left(y^\pi\right)_{\pi(a)\pi(B)} = y_{aB} \quad \text{for all } B \in \mathcal{S}_A \text{ and all } a \in B.$$

Observe that $x^\pi \in X^*$ and $y^\pi \in Y^*$.

*Neutrality* For all $x \in X^*$ and $\pi \in \Pi$, $f(x^\pi) = (f(x))^\pi$.

Theorem 1. *There exists an extension rule satisfying stochastic rationalizability and neutrality.*

Proof. For any $x \in X^*$, define a *carrier* of $x$ to be an inclusion-minimal set $\mathcal{D} \subseteq \mathcal{P}$ such that $x \in co\{x^*(P) | P \in \mathcal{D}\}$. By minimality, the points $x^*(P)(P \in \mathcal{D})$ are affinely independent[8] and there exists a unique collection of strictly positive weights $\alpha_{\mathcal{D}}(P)(P \in \mathcal{D})$ summing up to one such that

$$x = \sum_{P \in \mathcal{D}} \alpha_{\mathcal{D}}(P) x^*(P).$$

The random choice function

$$f_{\mathcal{D}}(x) = \sum_{P \in \mathcal{D}} \alpha_{\mathcal{D}}(P) y^*(P)$$

[8]By definition, $x^*(P) \in \{0, 1\}^{\mathcal{B}_A}$ for each $P \in \mathcal{P}$. A collection of points $x^1, \dots, x^K \in \{0, 1\}^{\mathcal{B}_A}$ are affinely independent if $[\sum_{k=1}^K \alpha_k x^k = 0 \text{ and } \sum_{k=1}^K \alpha_k = 0] \Rightarrow [\alpha_1 = \cdots = \alpha_K = 0]$.

is a stochastically rationalizable extension of $x$. Let $\boldsymbol{D}_x$ denote the set of carriers of $x$ and define

$$\overline{f}(x) = \frac{1}{|\boldsymbol{D}_x|} \sum_{\mathcal{D} \in \boldsymbol{D}_x} f_{\mathcal{D}}(x).$$

The extension rule $\overline{f} : X^* \to Y$ satisfies stochastic rationalizability and neutrality. □

There exist other stochastically rationalizable and neutral extension rules than the rule $\overline{f}$ defined in the proof above. For each $x \in X^*$, define $\Delta_x = \{\alpha \in \Delta(\mathcal{P}) | x = x^*(\alpha)\}$ and call the elements of $\Delta_x$ *decompositions* of $x$: these are the probability distributions that generate $x$. The set $\Delta_x$ is a convex polytope included in $\Delta(\mathcal{P})$. Its extreme points are the decompositions whose support is a carrier of $x$, i.e., $\Delta_x = co\{\alpha_{\mathcal{D}} | \mathcal{D} \in \boldsymbol{D}_x\}$.

Perhaps the most natural stochastically rationalizable and neutral extension rule is

$$\overline{\overline{f}}(x) = \int_{\Delta_x} y^*(\alpha)\, d\mu(\alpha),$$

where $\mu$ is the uniform probability measure on $\Delta_x$. This rule assigns to $x$ the uniform average of the random dictatorships generated by all the possible decompositions of $x$. By contrast, $\overline{f}(x)$ is the uniform average of the random dictatorships generated by the extreme decompositions of $x$. The rule $\overline{\overline{f}}$ is difficult to compute and it is not obvious whether it differs from $\overline{f}$.

For yet another example, let $L$ denote the leximax ordering[9] on $\Delta(\mathcal{P})$, let $\alpha_x^L$ be the unique minimal element of $L$ in the compact and convex set $\Delta_x$, and define $f^L(x) = y^*(\alpha_x^L)$. It is easy to see that the extension rule $f^L$ satisfies the two axioms in Theorem 1. We show in the Appendix that it differs from $\overline{f}$.

### 4.2 *Incompatibilities*

This subsection shows that several natural axioms are incompatible with (weak versions of) stochastic rationalizability. We begin with a property suggested by the algebraic structure of the sets $X^*$ and $Y$.

*Linearity* The map $f : X^* \to Y$ is an affine function, i.e.,

$$f\big(\lambda x + (1-\lambda)x'\big) = \lambda f(x) + (1-\lambda) f\big(x'\big) \tag{5}$$

for all $x, x' \in X^*$ and $\lambda \in [0, 1]$.

The fractional tournament $\lambda x + (1-\lambda)x'$ is generated by an electorate composed of two constituencies: one containing a fraction $\lambda$ of the total population and generating the fractional tournament $x$, the other containing a fraction $1-\lambda$ of the population and generating the fractional tournament $x'$. In the aggregate electorate, linearity

[9]For any $\alpha \in \Delta(\mathcal{P})$, let $\widehat{\alpha}$ be the vector obtained from $\alpha$ by arranging its coordinates in nondecreasing order, i.e., $\widehat{\alpha} = (\widehat{\alpha}(1), \ldots, \widehat{\alpha}(m!)) = (\alpha(\pi(1)), \ldots, \alpha(\pi(m!)))$ for any bijection $\pi : \{1, \ldots, m!\} \to \mathcal{P}$ such that $\alpha(\pi(1)) \leq \cdots \leq \alpha(\pi(m!))$. The leximax ordering $L$ on $\Delta(\mathcal{P})$ is defined by letting $\alpha L \beta$ if and only if either $\widehat{\alpha} = \widehat{\beta}$ or there exists $k \in \{1, \ldots, m!\}$ such that $\widehat{\alpha}_k > \widehat{\beta}_k$ and $\widehat{\alpha}_{k'} = \widehat{\beta}_{k'}$ for all $k' > k$.

recommends to use the random choice function employed in each constituency with a probability equal to the weight of that constituency in the total population.[10]

An important motivation for this axiom comes from the linearity of the random utility model itself. That is, the function $y^* : \Delta(\mathcal{P}) \to Y$ defined by (2) satisfies

$$y^*\big(\lambda\alpha + (1-\lambda)\alpha'\big) = \lambda y^*(\alpha) + (1-\lambda)y^*\big(\alpha'\big) \tag{6}$$

for all $\alpha, \alpha' \in \Delta(\mathcal{P})$, and $\lambda \in [0, 1]$. It follows that on any subset of $X^*$ admitting a unique stochastically rational extension rule, that extension rule *is* an affine function. Formally, let $\mathcal{D} \subseteq \mathcal{P}$ be a domain of preferences such that $x^*(P)(P \in \mathcal{D})$ are affinely independent, and define $X^*_{\mathcal{D}} := co\{x^*(P)|P \in \mathcal{D}\}$ and $Y^*_{\mathcal{D}} := co\{y^*(P)|P \in \mathcal{D}\}$. Since for any $x \in X^*_{\mathcal{D}}$ there is a unique probability distribution $\alpha$ on $\mathcal{D}$ such that $x = \sum_{P\in\mathcal{D}} \alpha(P)x^*(P)$, the function $x \mapsto \sum_{P\in\mathcal{D}} \alpha(P)y^*(P)$ is the unique admissible extension rule on $X^*_{\mathcal{D}}$, i.e., the only function $f : X^*_{\mathcal{D}} \to Y^*_{\mathcal{D}}$ such that $f(x)$ extends $x$ for every $x \in X^*_{\mathcal{D}}$. Because of (6), it is an affine function.

Linearity embodies a natural idea of "betweenness preservation": since $\alpha x + (1 - \alpha)x'$ describes a society that lies between those described by $x$ and $x'$, the random choice function associated with $\alpha x + (1-\alpha)x'$ should lie between those associated with $x$ and $x'$. Note that the proportional Borda rule, for instance, satisfies linearity:

$$\begin{aligned} f^{\text{Borda}}_{aB}\big(\lambda x + (1-\lambda)x'\big) &= \frac{\sum\limits_{c\in B\setminus a} \big(\lambda x_{ac} + (1-\lambda)x'_{ac}\big)}{\binom{|B|}{2}} \\ &= \lambda \frac{\sum\limits_{c\in B\setminus a} x_{ac}}{\binom{|B|}{2}} + (1-\lambda)\frac{\sum\limits_{c\in B\setminus a} x'_{ac}}{\binom{|B|}{2}} \\ &= \lambda f^{\text{Borda}}_{aB}(x) + (1-\lambda) f^{\text{Borda}}_{aB}\big(x'\big) \end{aligned}$$

for all $x, x' \in X^*$, $\lambda \in [0, 1]$, $B \in \mathcal{S}_A$, and $a \in B$.

Of course, convex combinations express a specific, "cardinal" form of betweenness. A more abstract, "ordinal" version of betweenness preservation may be defined by using only the order structure of the sets $X^*$ and $Y$. For any $x, x', x'' \in X^*$, and $y, y', y'' \in Y$, write $x'' \in [x, x']$ whenever $\min\{x_{ab}, x'_{ab}\} \leq x''_{ab} \leq \max\{x_{ab}, x'_{ab}\}$ for all $(a, b) \in \mathcal{B}_A$ and $y'' \in [y, y']$ whenever $\min\{y_{aB}, y'_{aB}\} \leq y''_{aB} \leq \max\{y_{aB}, y'_{aB}\}$ for all $B \in \mathcal{S}_A$ and $a \in B$.

*Betweenness preservation* For all $x, x', x'' \in X^*$, $[x'' \in [x, x']] \Rightarrow [f(x'') \in [f(x), f(x')]]$.

[10]This can be regarded as a strengthening of Young's (1975) reinforcement axiom. In our setting, reinforcement stipulates that for all $x, x' \in X^*$, $[f(x) = f(x')] \Rightarrow [f(\lambda x + (1-\lambda)x') = f(x)$ for all $\lambda \in [0, 1]]$. Notice that every extension rule $f$ trivially satisfies reinforcement since $[f(x) = f(x')] \Rightarrow [x = x']$ for all $x \in X^*$.

Linearity and betweenness preservation are very demanding. Both are incompatible with stochastic rationalizability. In fact, they conflict with the following very mild consequence of the latter axiom.

*Rationalizability for unanimous tournaments* For all $P \in \mathcal{P}$, $f(x^*(P)) \in Y^*$.

Observe that this axiom is also a consequence of the unanimity principle: since $x^*(P)$ arises from a population of voters sharing the common preference ordering $P$, $f(x^*(P))$ should select the best feasible alternative according to $P$, i.e., $f(x^*(P)) = y^*(P)$. This implies $f(x^*(P)) \in Y^*$.

Theorem 2. *If $m \geq 3$, no extension rule satisfies (i) rationalizability for unanimous tournaments and (ii) linearity or betweenness preservation.*

Proof. Let us first show that rationalizability for unanimous tournaments is incompatible with linearity. Suppose, by way of contradiction, that $f$ satisfies both axioms. Focusing on the case $A = \{1, 2, 3\}$, consider again the decompositions (3) and (4) of the fractional tournament $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Since the only stochastically rationalizable extension of a unanimous tournament $x^*(P)$ is $y^*(P)$, applying linearity and rationalizability for unanimous tournaments to (3) yields $f(x) = \frac{1}{2}f(x^*(123)) + \frac{1}{2}f(x^*(321)) = \frac{1}{2}y^*(123) + \frac{1}{2}y^*(321)$. Likewise, applying the axioms to (4) yields $f(x) = \frac{1}{2}f(x^*(132)) + \frac{1}{2}f(x^*(231)) = \frac{1}{2}y^*(132) + \frac{1}{2}y^*(231)$. Since $\frac{1}{2}y^*_{\{1,2,3\}}(123) + \frac{1}{2}y^*_{\{1,2,3\}}(321) = (\frac{1}{2}, 0, \frac{1}{2})$ and $\frac{1}{2}y^*_{\{1,2,3\}}(132) + \frac{1}{2}y^*_{\{1,2,3\}}(231)) = (\frac{1}{2}, \frac{1}{2}, 0)$, we conclude that $f(x) \neq f(x)$, which is impossible.

The incompatibility between rationalizability for unanimous tournaments and betweenness preservation is even more radical: it holds on the subset of unanimous tournaments. Indeed, $x^*(213) = (0, 1, 0) \in [x^*(123), x^*(321)] = [(1, 1, 0), (0, 0, 1)]$ but $y^*(213) \notin [y^*(123), y^*(321)]$ since $y^*_{\{1,2,3\}}(213) = (0, 1, 0) \notin [y^*_{\{1,2,3\}}(123), y^*_{\{1,2,3\}}(321)] = [(1, 0, 0), (0, 0, 1)]$. □

We conclude this section by studying a variant of Arrow's independence of irrelevant alternatives stipulating that the probability of choosing an alternative from an agenda should only depend on the restriction of the fractional tournament to that agenda. For any $B \in \mathcal{S}_A$ and $x \in X^*$, let $\mathcal{B}_B = \{(a, b) \in B \times B | a \neq b\}$ and let $x_B = (x_{ab})_{(a,b)\in\mathcal{B}(B)}$ denote the restriction of the fractional tournament $x$ to the alternatives in $B$.

*Independence of irrelevant comparisons* For all $x, x' \in X^*$ and $B \in \mathcal{S}_A$,

$$[x_B = x'_B] \quad \Rightarrow \quad [f_B(x) = f_B(x')]. \tag{7}$$

The motivation is the same as for Arrow's axiom. Young (1995) summarizes it as follows:

> "There are at least two reasons why this is desirable from a practical standpoint. First, if it does not hold, then it is possible to manipulate the outcome by introducing extraneous alternatives. [...] Second, independence allows the electorate to make sensible decisions within a restricted range of choices without worrying about the universe of all possible choices. It is desirable to know, for example, that the relative ranking of candidates for political office would not be changed if purely hypothetical candidates were included on the ballot."

The following extension rule shows that independence of irrelevant comparisons is compatible with rationalizability for unanimous tournaments: for all $B \in \mathcal{S}_A$ and $a \in B$, let

$$f_{aB}(x) = \begin{cases} y^*_{aB}(P) & \text{if } x_B = x^*_B(P) \text{ for some } P \in \mathcal{P}, \\ \dfrac{1}{\binom{|B|}{2}} \sum_{b \in B \setminus a} x_{ab} & \text{otherwise.} \end{cases}$$

This rule is well-defined because $y^*_B(P)$ is identical for all $P \in \mathcal{P}$ such that $x_B = x^*_B(P)$.

Unfortunately, independence of irrelevant comparisons clashes with another elementary consequence of stochastic rationalizability known as agenda monotonicity. A random choice function $y$ is *agenda-monotonic* if the probability of choosing an alternative from an agenda does not increase when that agenda expands: $y_{aB} \geq y_{aB'}$ whenever $a \in B \subseteq B' \subseteq A$. This property is the natural counterpart of Chernoff's (1954) axiom (or Sen's (1970) condition $\alpha$) for deterministic choice correspondences, which requires that $a$ should not be chosen from $B'$ if it is not chosen from $B$. Let $Y^{\text{mon}}$ denote the set of agenda-monotonic random choice functions.

*Agenda monotonicity* $f(X^*) \subseteq Y^{\text{mon}}$.

It is well known that $Y^*$ is a proper subset of $Y^{\text{mon}}$ when $m \geq 4$. In fact, Falmagne's (1978) classic characterization of the rationalizable random choice functions shows that agenda monotonicity is much weaker than stochastic rationalizability.

THEOREM 3. *If $m \geq 4$, no extension rule satisfies agenda monotonicity and independence of irrelevant comparisons.*

PROOF. It is enough to establish the incompatibility when $m = 4$. Suppose, contrary to the claim, that $f : X^* \to Y$ satisfies agenda monotonicity and independence of irrelevant comparisons. Consider the fractional tournament

$$x = \begin{bmatrix} - & 0.5 & 0.5 & 0.6 \\ 0.5 & - & 0.5 & 0.1 \\ 0.5 & 0.5 & - & 0.1 \\ 0.4 & 0.9 & 0.9 & - \end{bmatrix}.$$

To check that $x \in X^*$, note that $x$ is generated by the following (10-voters) profile[11] $P = (P^1, \ldots, P^{10})$:

| $P^1$ | $P^2$ | $P^3, P^4$ | $P^5, P^6$ | $P^7, P^8$ | $P^9, P^{10}$ |
|---|---|---|---|---|---|
| 3 | 2 | 1 | 1 | 4 | 4 |
| 1 | 1 | 4 | 4 | 3 | 2 |
| 4 | 4 | 3 | 2 | 2 | 3 |
| 2 | 3 | 2 | 3 | 1 | 1 |

[11]Alternatively, it suffices to check that $x$ satisfies the triangle inequalities; See footnote 5.

Since $f$ satisfies agenda monotonicity,

$$f_{2\{1,2,3,4\}}(x) \le x_{24} = 0.1,$$
$$f_{3\{1,2,3,4\}}(x) \le x_{34} = 0.1,$$
$$f_{4\{1,2,3,4\}}(x) \le x_{41} = 0.4.$$

Since $\sum_{a=1}^{4} f_{a\{1,2,3,4\}}(x) = 1$, the three inequalities above imply $f_{1\{1,2,3,4\}}(x) \ge 0.4$, hence, by agenda monotonicity,

$$f_{1\{1,2,3\}}(x) \ge 0.4. \tag{8}$$

Next, applying the same argument to the fractional tournaments

$$x' = \begin{bmatrix} - & 0.5 & 0.5 & 0.1 \\ 0.5 & - & 0.5 & 0.6 \\ 0.5 & 0.5 & - & 0.1 \\ 0.9 & 0.4 & 0.9 & - \end{bmatrix}, \qquad x'' = \begin{bmatrix} - & 0.5 & 0.5 & 0.1 \\ 0.5 & - & 0.5 & 0.1 \\ 0.5 & 0.5 & - & 0.6 \\ 0.9 & 0.9 & 0.4 & - \end{bmatrix},$$

leads to the inequalities

$$f_{2\{1,2,3\}}(x') \ge 0.4, \tag{9}$$
$$f_{3\{1,2,3\}}(x'') \ge 0.4. \tag{10}$$

But since $x_{ab} = x'_{ab} = x''_{ab}$ for all $(a, b) \in \mathcal{B}_{\{1,2,3\}}$, independence of irrelevant comparisons implies $f_{\{1,2,3\}}(x) = f_{\{1,2,3\}}(x') = f_{\{1,2,3\}}(x'')$. Hence, (8), (9), and (10) imply $\sum_{a=1}^{3} f_{a\{1,2,3\}}(x) \ge 1.2$, which is impossible. □

Two remarks are in order.

(1) Theorem 3 follows from a result of Pattanaik and Peleg (1986).[12] In Pattanaik and Peleg's setting, there is a given finite set of voters, $N = \{1, \dots, n\}$, $n \ge 2$, and a *probabilistic voting procedure (PVP)* is a function $g : \mathcal{P}^N \to Y$. Say that such a PVP $g$ satisfies *agenda monotonicity** if

$$g(P^N) \subseteq Y^{\text{mon}}$$

for every preference profile $P^N \in \mathcal{P}^N$. For any $B \in \mathcal{S}_A$ and $P^N \in \mathcal{P}^N$, let $P^N_B$ denote the restriction of $P^N$ to $B$ and say that $g$ satisfies *independence of irrelevant comparisons** if

$$[P^N_B = \overline{P}^N_B] \quad \Rightarrow \quad [g_B(P^N) = g_B(\overline{P}^N)]$$

for all $P^N, \overline{P}^N \in \mathcal{P}^N$ and all $B \in \mathcal{S}_A$. Finally, call $g$ *Paretian* if

$$[g_{aB}(P^N) > 0] \quad \Rightarrow \quad [\nexists b \in B \text{ such that } bP^i a \text{ for all } i \in N]$$

[12] I am grateful to a referee for pointing out this connection and suggesting the argument that follows.

for all $P^N \in \mathcal{P}^N$, $B \in \mathcal{S}_A$, and $a \in B$. Let $\Delta(N) = \{\beta \in [0,1]^N | \sum_{i \in N} \beta(i) = 1\}$ be the set of probability distributions on the set of voters $N$. Pattanaik and Peleg (1986) prove the following result.[13]

Theorem (Pattanaik and Peleg). *If $m \geq 4$ and $g : \mathcal{P}^N \to Y$ is a Paretian PVP satisfying agenda monotonicity* and independence of irrelevant comparisons*, there exists $\beta \in \Delta(N)$ such that*

$$g_B(P^N) = \sum_{i \in N} \beta(i) y^*_B(P^i)$$

*for all $P^N \in \mathcal{P}^N$ and all $B \in \mathcal{S}_A$ such that $B \neq A$.*

This theorem implies Theorem 3. To see why, let $m \geq 4$ and suppose, by way of contradiction, that $f : X^* \to Y$ is an extension rule satisfying agenda monotonicity and independence of irrelevant comparisons. Let $N = \{1, 2, \ldots, 10\}$. For every preference profile $P^N = (P^1, \ldots, P^{10}) \in \mathcal{P}^N$, denote by $\alpha_{P^N} \in \Delta(\mathcal{P})$ the probability distribution assigning to each linear ordering the fraction of voters in $N$ whose preference coincides with that ordering, i.e.,

$$\alpha_{P^N}(P) = \frac{|\{i \in N | P^i = P\}|}{|N|}$$

for all $P \in \mathcal{P}$. Define the PVP $g : \mathcal{P}^N \to Y$ by

$$g(P^N) = f(x^*(\alpha_{P^N})) \tag{11}$$

for all $P^N \in \mathcal{P}^N$, where $x^*(\alpha_{P^N})$ is the fractional tournament generated by $\alpha_{P^N}$, as defined in (1).

It is straightforward to check that $g$ satisfies agenda monotonicity* and independence of irrelevant comparisons*. To check that $g$ is Paretian, let $P^N \in \mathcal{P}^N$, $B \in \mathcal{S}_A$, $a, b \in B$, and suppose that $bP^i a$ for all $i \in N$. By definition, $x^*_{ab}(\alpha_{P^N}) = 0$. By (11) and because $f$ is an extension rule, $g_{a\{a,b\}}(P^N) = f_{a\{a,b\}}(x^*(\alpha_{P^N})) = x^*_{ab}(\alpha_{P^N}) = 0$. Since $g$ satisfies agenda monotonicity*, $g_{aB}(P^N) \leq g_{a\{a,b\}}(P^N)$, hence $g_{aB}(P^N) = 0$.

By Pattanaik and Peleg's theorem, there exists $\beta \in \Delta(N)$ such that $g_B(P^N) = \sum_{i \in N} \beta(i) y^*_B(P^i)$ for all $P^N \in \mathcal{P}^N$ and all $B \in \mathcal{S}_A \setminus \{A\}$. Next, note that $g$ is *anonymous* in the traditional sense: for every $(P^1, \ldots, P^{10}) \in \mathcal{P}^N$ and every bijection $\sigma$ from $N$ to $N$, $\alpha_{(P^{\sigma(1)}, \ldots, P^{\sigma(10)})} = \alpha_{(P^1, \ldots, P^{10})} \Rightarrow x^*(\alpha_{(P^{\sigma(1)}, \ldots, P^{\sigma(10)})}) = x^*(\alpha_{(P^1, \ldots, P^{10})}) \Rightarrow g(P^{\sigma(1)}, \ldots, P^{\sigma(10)}) = g(P^1, \ldots, P^{10})$. Because $g$ is anonymous, it is easy to see that $\beta$ must be uniform, i.e.,

$$g_B(P^N) = \frac{1}{10} \sum_{i=1}^{10} y^*_B(P^i) \tag{12}$$

for all $P^N \in \mathcal{P}^N$ and all $B \in \mathcal{S}_A \setminus \{A\}$.

[13] This is Pattanaik and Peleg's Theorem 4.11, restated using our terminology and notation.

Applying (12) to the preference profile $P^N = (P^1, \dots, P^{10})$ defined in the proof of Theorem 3 and $B = \{1, 2, 3\}$ yields

$$g_{\{1,2,3\}}(P^N) = (0.4, 0.3, 0.3).$$

Similarly, applying (12) to the profile $\overline{P}_N$ given by

| $\overline{P}^1$ | $\overline{P}^2$ | $\overline{P}^3, \overline{P}^4, \overline{P}^5$ | $\overline{P}^6$ | $\overline{P}^7$ | $\overline{P}^8, \overline{P}^9, \overline{P}^{10}$ |
|---|---|---|---|---|---|
| 3 | 2 | 1 | 1 | 4 | 4 |
| 1 | 1 | 4 | 4 | 3 | 2 |
| 4 | 4 | 3 | 2 | 2 | 3 |
| 2 | 3 | 2 | 3 | 1 | 1 |

and $B = \{1, 2, 3\}$ yields

$$g_{\{1,2,3\}}(\overline{P}^N) = (0.4, 0.4, 0.2).$$

Thus, $g(P^N) \neq g(\overline{P}^N)$. Check, however, that $x^*(\alpha_{P^N}) = x^*(\alpha_{\overline{P}^N})$. It follows from (11) that $g(P^N) = g(\overline{P}^N)$, a contradiction.

(2) Moulin (1986) proves a variant of Theorem 3 for a rule $\varphi$ associating a deterministic choice correspondence to every pure tournament (i.e., every fractional tournament belonging to $\{0, 1\}^{\mathcal{B}_A}$). He shows that no such rule satisfies Condorcet consistency, Arrow's IIA, and Chernoff. Condorcet consistency can be weakened to the requirement that $\varphi$ is an extension rule, i.e., alternative $a$ is the unique choice from $\{a, b\}$ whenever $a$ beats $b$. Arrow's IIA is the deterministic counterpart of independence of irrelevant comparisons. As mentioned earlier, Chernoff requires that an alternative rejected from an agenda be rejected from any superset of that agenda, and our axiom of agenda monotonicity may be regarded as a very weak random version of Chernoff. Moulin's result holds if $m \geq 3$ while Theorem 3 requires $m \geq 4$.[14]

## 5. Sequentially binary domains

In this section, we explore preference domains generating fractional tournaments that have a unique stochastically rationalizable extension with support in that domain. If voters' preferences belong to such a domain, the information revealed by the associated fractional tournament to a decision maker who subscribes to the stochastic rationalizability axiom fully pins down her random choices from all agendas. The approach adopted here thus complements the axiomatic analysis of Section 4 by identifying conditions under which the search for axioms supplementing stochastic rationalizability is unnecessary. We note that the theory of majority voting is similarly divided into two strands: the core of the theory, which proposes methods for making deterministic choices extending an arbitrary majority tournament, is complemented by a domain-restriction literature studying conditions ensuring the transitivity of the majority tournament. See, for instance, Chapter 10 of Moulin (1988).

[14]When $m = 3$, every stochastically rationalizable extension rule $f$ satisfies independence of irrelevant comparisons (because the requirement $f_{a\{a,b\}}(x) = x_{ab}$ implies $f_{a\{a,b\}}(x) = f_{a\{a,b\}}(x')$ for all $x, x'$ such that $x_{\{a,b\}} = x'_{\{a,b\}}$) and agenda monotonicity (because that axiom is implied by stochastic rationalizability).

A *domain* (on $A$) is a set $\mathcal{D} \subseteq \mathcal{P}$. In some contexts, the collective decision maker may know that the support of the probability distribution $\alpha \in \Delta(\mathcal{P})$ generating $x$ is included in a given domain $\mathcal{D}$ of admissible preferences, and this may suffice to guarantee that $x$ has a unique admissible stochastically rationalizable extension.

A trivial case occurs when the fractional tournaments $x^*(P)$ corresponding to the orderings $P$ belonging to the domain $\mathcal{D}$ are affinely independent. The distribution $\alpha$ itself is then unique. But since the dimension of $X^*$ is $m(m-1)/2$, the domain $\mathcal{D}$ contains at most $m(m-1)/2+1$ orderings. The single-crossing domains discussed in Apesteguia, Ballester, and Lu (2017) are an example.

This section studies a class of domains that we call *sequentially binary*. A sequentially binary domain contains $2^{m-1}$ orderings. Each ordering results from $m-1$ successive binary choices. The first binary choice determines the worst alternative. The second binary choice, which depends upon the outcome of the first, determines the second-worst alternative. The last binary choice is conditional upon the outcome of the $m-2$ preceding choices and determines the second-best alternative, hence also the first-best alternative. There are restrictions tying the successive choices, which will be described shortly.

Although a fractional tournament $x$ may be generated by several probability distributions with support in a given sequentially binary domain, we will show that all such distributions generate the same random choice function. This means that the random choices from agendas of all sizes are completely determined by the requirement of stochastic rationalizability.

For each $m \in \mathbb{N}$, let $S^m = \{0, 1\}^m$ and $S^{(m)} = \bigcup_{k=1}^m S^k$. Write $S = \bigcup_{k \in \mathbb{N}} S^k$. An element of $S^m$ is a sequence $s = (s_1, \dots, s_m)$ of $m$ numbers in $\{0, 1\}$. We call it *binary* and refer to $m$ as its *length*. An element of $S^{(m)}$ is a binary sequence of length at most $m$, and an element of $S$ is a binary sequence of finite length. For convenience, we denote by $s_0$ the "empty sequence" (of zero length) and define $S_0^{(m)} = S^{(m)} \cup \{s_0\}$ and $S_0 = S \cup \{s_0\}$. We say that $s = (s_1, \dots, s_k)$ *precedes* $s' = (s'_1, \dots, s'_{k'})$ (or, equivalently, $s'$ *follows* $s$) if $k < k'$ and $s = (s'_1, \dots, s'_k)$, which we write $s \prec s'$. By convention, $s_0 \prec s$ for every $s \in S$. The *direct followers* of $s = (s_1, \dots, s_k) \in S$ are $(s, 0) := (s_1, \dots, s_k, 0)$ and $(s, 1) := (s_1, \dots, s_k, 1)$. By convention, the direct followers of $s_0$ are the two sequences $(0)$ and $(1)$.

One can think of $(S_0^{(m-1)}, \prec)$ as a binary tree whose nodes are the sequences $s \in S_0^{(m-1)}$. The tree is rooted at $s_0$ and its terminal nodes are the binary sequences of length $m-1$. Figure 1 depicts the case $m=3$; an edge is drawn between each nonterminal node and its direct followers.

Definition 1. A *selection function* (into $A$) is a function $g : S^{(m-1)} \to A$ such that

$$g(s) \neq g(s') \quad \text{for all } s, s' \in S^{(m-1)} \text{ such that } s \prec s', \tag{13}$$

$$g(s, 0) \neq g(s, 1) \quad \text{for all } s \in S_0^{(m-2)}. \tag{14}$$

We will again write an ordering $P \in \mathcal{P}$ by listing the $m$ alternatives in $A$ from best to worst: thus, $P = a_1 a_2 \cdots a_m$ is the ordering according to which $a_1$ is the best alternative in

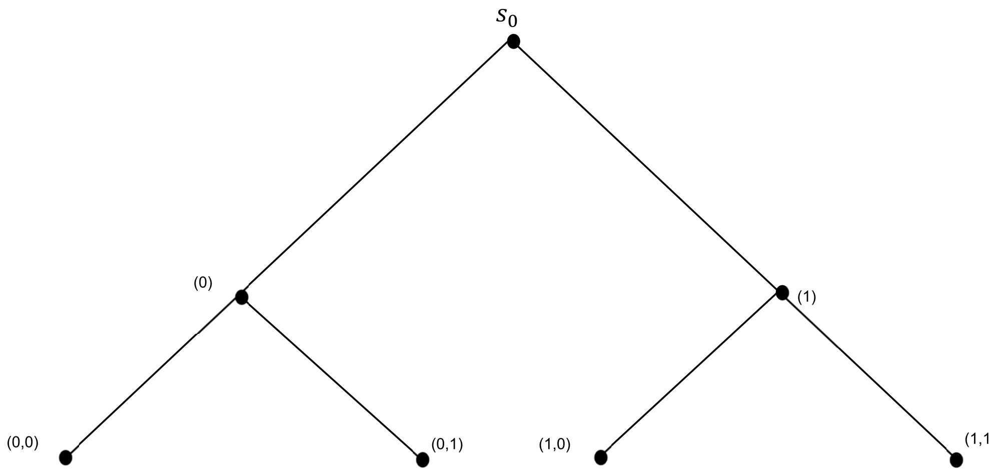


FIGURE 1. The binary tree $(S_0^{(2)}, \prec)$

$A$, $a_2$ is the second best, and so on. For each $s = (s_1, \dots, s_{m-1}) \in S^{m-1}$, define $P_g(s) \in \mathcal{P}$ by

$$P_g(s) := a_g(s) g(s_1, \dots, s_{m-1}) \cdots g(s_1, s_2) g(s_1),$$

where $a_g(s)$ is the unique alternative in $A \setminus \{g(s_1, \dots, s_{m-1}), \dots, g(s_1, s_2), g(s_1)\}$. This is well-defined because condition (13) ensures that $g(s_1, \dots, s_{m-1}), \dots, g(s_1, s_2), g(s_1)$ are distinct alternatives. One can think of $P_g(s)$ as constructed "from bottom to top" by filling up the successive ranks: at each node $(s_1, \dots, s_k) \prec s$, the two alternatives $g(s_1, \dots, s_k, 0)$ and $g(s_1, \dots, s_k, 1)$ are offered to fill rank $m - k$, and the successive choices determine $P_g(s)$.

Condition (14) implies that $P_g(s) \neq P_g(s')$ if $s \neq s'$. It follows that

$$\mathcal{D}_g := \{P_g(s) | s \in S^{m-1}\}$$

contains exactly $2^{m-1}$ distinct orderings on $A$. To get a grasp on the domain $\mathcal{D}_g$, it may be helpful to identify the orderings that do *not* belong to it. All orderings for which the worst alternative is neither $g((0))$ nor $g((1))$ are excluded. Among the orderings whose worst alternative is $g((0))$, all those for which the second-worst alternative is neither $g((0, 0))$ nor $g((0, 1))$ are ruled out, and so on.

The sequentially binary domains are generated by a subclass of selection functions that we now describe. For any $s, s' \in S$, let us write $s \precsim s'$ if $s \prec s'$ or $s = s'$, and let $WP(s) = \{s' \in S | s' \precsim s\}$ be the set of binary sequences that *weakly precede* $s$. The *twin* of $s = (s_1, \dots, s_k) \in S$ is the sequence $tw(s) = (s_1, \dots, s_{k-1}, s'_k) \in S$ such that $s'_k \neq s_k$. For each $s \in S_0^{(m-2)}$, let $O_g(s) = \{g(s, 0), g(s, 1)\}$. This may be interpreted as the "option set" generated by $g$ at $s$: it contains the two alternatives competing to fill the rank open at $s$.

DEFINITION 2. A selection function $g$ into $A$ is *consistent* if

$$g(s) \in O_g(tw(s)) \quad \text{for all } s \in S^{(m-2)} \tag{15}$$

and

$$O_g(s) = O_g(s') \quad \text{for all } s, s' \in S^{(m-2)} \text{ such that } g(WP(s)) = g(WP(s')). \tag{16}$$

Condition (15) says that an alternative offered to fill the rank open at a node must be offered again at the node reached by rejecting that alternative. Condition (16) says that the pair of alternatives offered at a node $s$ may only depend upon the set of alternatives that were selected at the nodes preceding $s$, but not upon the order in which they were selected.

DEFINITION 3. Let $G_A$ be the set of consistent selection functions into $A$. A domain $\mathcal{D} \subseteq \mathcal{P}$ is *sequentially binary* if $\mathcal{D} = \mathcal{D}_g$ for some $g \in G_A$.

When $m = 3$, it is easy to see that a domain is sequentially binary if and only if it is a maximal single-peaked domain in the sense of Black (1948).[15]

Two examples of sequentially binary domains are illustrated in Figure 2 for the case $m = 4$. For each node $s$ other than the root, $g(s)$ is indicated next to $s$. For each terminal node $s$, the ordering $P_g(s)$ is recorded below $s$ by listing the alternatives from best to worst.

At each node in Figure 2(a), the choice is between the smallest and the largest alternatives that remain to be ranked. The resulting domain $\mathcal{D}_g$ contains the eight orderings that are *single-peaked* with respect to the ordering 1234. The domain in Figure 2(b) is inspired by the *successive elimination* voting rule (Moulin (1988, p. 241)). In stage $k = 1, \dots, m-1$, the largest two alternatives among those remaining to be ranked are paired and compete for rank $m - k + 1$. Up to a relabeling of the alternatives, these are the only two sequentially binary domains when $m = 4$.

Figure 3 depicts two domains that are *not* sequentially binary.

In Figure 3(a), the underlying selection function violates condition (15). Indeed, $g((1)) = 4 \notin O_g((0)) = \{1, 2\}$: alternative 4 is offered to fill rank 4 at the root but is not offered to fill rank 3 at the sequence $s = (0)$ that is reached by rejecting 4. Indeed, $g(WP((0, 1))) = g(WP((1, 0))) = \{1, 5\}$ but $O((0, 1)) = \{2, 3\} \neq O((1, 0)) = \{2, 4\}$. Note that condition (15) implies a form of connectedness of $\mathcal{D}_g$: if an alternative is ranked last by an ordering in the domain, it is ranked at any possible rank by an ordering in the domain. Condition (16) is vacuous when $m \leq 4$. In Figure 3(b), $m = 5$ and the selection function $g$ satisfies condition (15) but not (16). Indeed, $g(WP((0, 1))) = g(WP((1, 0))) = \{1, 5\}$ but $O((0, 1)) = \{2, 3\} \neq O((1, 0)) = \{2, 4\}$.

For any $\mathcal{D} \subseteq \mathcal{P}$, let $\Delta(\mathcal{D}) = \{\alpha \in [0, 1]^{\mathcal{D}} | \sum_{P \in \mathcal{D}} \alpha(P) = 1\}$ be the set of probability distributions on $\mathcal{D}$. For any $\alpha \in \Delta(\mathcal{D})$, we slightly abuse our earlier notation and write $x^*(\alpha) = \sum_{P \in \mathcal{D}} \alpha(P) x^*(P)$ and $y^*(\alpha) = \sum_{P \in \mathcal{D}} \alpha(P) y^*(P)$.

THEOREM 4. *If $\mathcal{D}$ is a sequentially binary domain and $\alpha, \alpha' \in \Delta(\mathcal{D})$, then* $[x^*(\alpha) = x^*(\alpha')] \Rightarrow [y^*(\alpha) = y^*(\alpha')]$.

[15] A domain $\mathcal{D} \subseteq \mathcal{P}$ is *single-peaked* with respect to a linear ordering $>$ on $A$ if $[b > a > \max_A P \text{ or } \max_A P > a > b] \Rightarrow [aPb]$ for all $P \in \mathcal{D}$, where $\max_A P$ denotes the best alternative in $A$ according to $P$. We call a domain single-peaked if it is single-peaked with respect to some linear ordering.

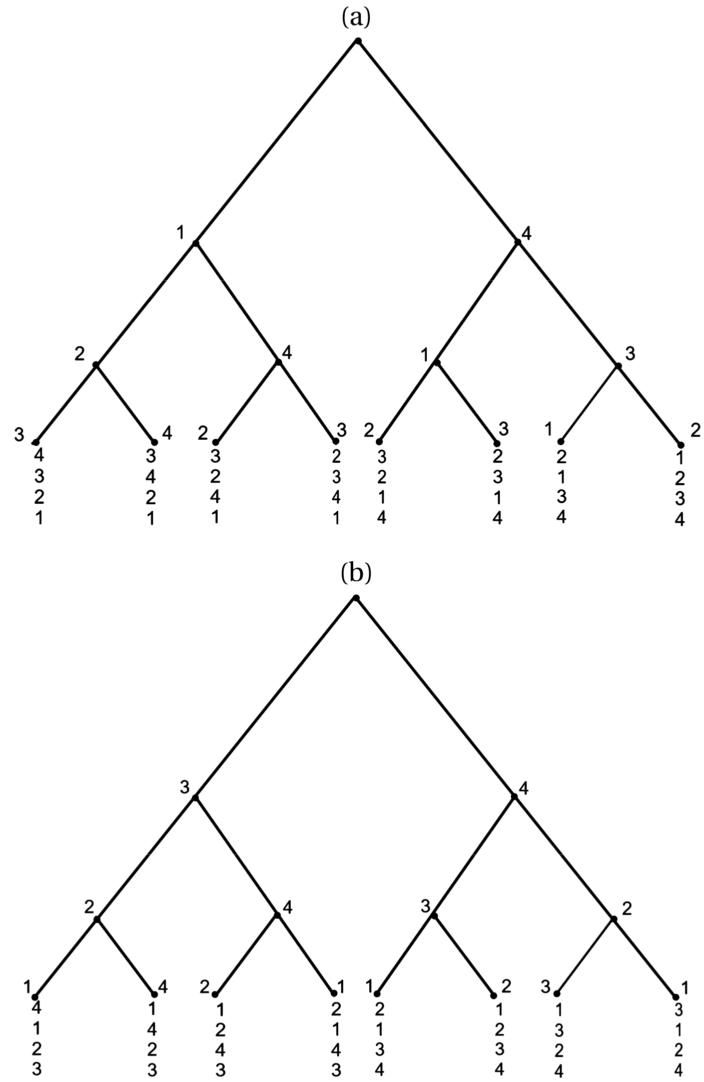


FIGURE 2. (a) A single-peaked domain. (b) A successive elimination domain.

In words: two distributions on a sequentially binary domain $\mathcal{D}$ that generate the same fractional tournament also generate the same random choice function. Thus, if a fractional tournament arises from a population of voters with preferences in $\mathcal{D}$, it has a unique admissible extension, namely a unique extension generated by a population of voters with preferences in $\mathcal{D}$.

To state this formally, recall our notation $X^*_{\mathcal{D}} = \{x^*(\alpha) | \alpha \in \Delta(\mathcal{D})\}$ and $Y^*_{\mathcal{D}} = \{y^*(\alpha) | \alpha \in \Delta(\mathcal{D})\}$. By an *extension rule on* $X^*_{\mathcal{D}}$, we mean a function $f : X^*_{\mathcal{D}} \to Y$ such that $f(x)$ extends $x$ for all $x \in X^*_{\mathcal{D}}$. We call $f$ *admissible* if $f(X^*_{\mathcal{D}}) \subseteq Y^*_{\mathcal{D}}$.

COROLLARY TO THEOREM 4. *If $\mathcal{D}$ is a sequentially binary domain, there is a unique admissible extension rule on $X^*_{\mathcal{D}}$.*

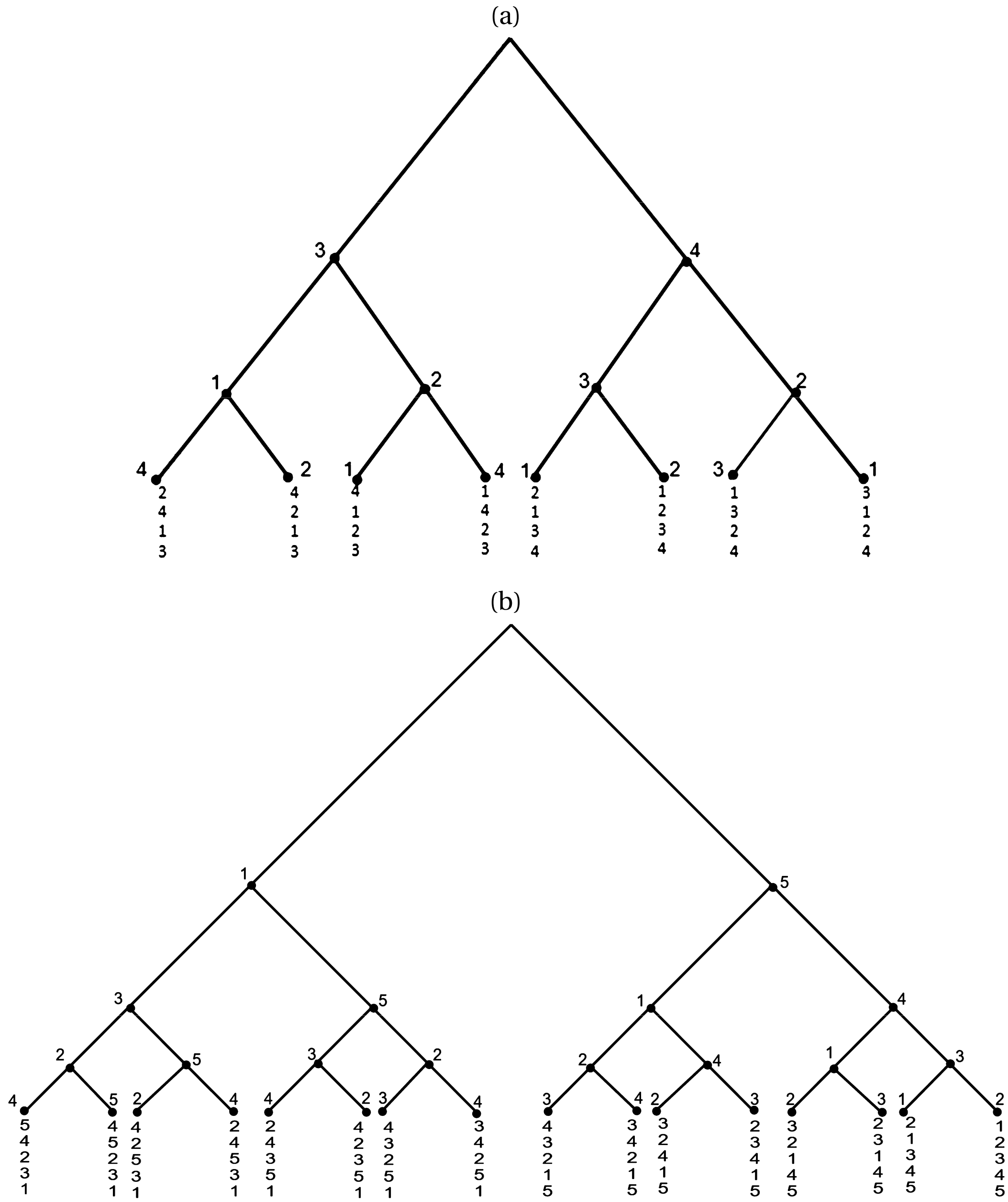


FIGURE 3. (a) A domain violating condition (16) (b) A domain violating condition (15).

Note that a fractional tournament $x \in X^*_{\mathcal{D}}$ may have multiple stochastically rationalizable extensions. What the corollary above states is that exactly one such extension is admissible. Consider the three-alternative case and let $\mathcal{D} = \{123, 213, 231, 321\}$. This is the single-peaked domain with respect to the natural ordering of the alternatives, an example of a sequentially binary domain. Clearly, $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}x^*(123) + \frac{1}{2}x^*(321) \in$

$X^*_{\mathcal{D}}$. Yet, $\frac{1}{2}y^*(123) + \frac{1}{2}y^*(321)$ and $\frac{1}{2}y^*(132) + \frac{1}{2}y^*(231)$ are two different stochastically rationalizable extensions of $x$. The first is admissible because $123, 321 \in \mathcal{D}$; the second is not because $132 \notin \mathcal{D}$.

The proof of Theorem 4 is given in the Appendix and the corollary follows directly from the theorem.

Two of the incompatibilities identified in Theorems 2 and 3 vanish on the fractional tournaments generated by preferences in a sequentially binary domain. Formally, say that an extension rule $f$ on $X^*_{\mathcal{D}}$ satisfies linearity on $X^*_{\mathcal{D}}$ if property (5) holds for all $x, x' \in X^*_{\mathcal{D}}$, and $\lambda \in [0, 1]$, and say that $f$ satisfies independence of irrelevant comparisons on $X^*_{\mathcal{D}}$ if property (7) holds for all $x, x' \in X^*_{\mathcal{D}}$, and $B \in \mathcal{S}_A$.

Theorem 5. *If $\mathcal{D}$ is a sequentially binary domain, the unique admissible extension rule $f$ on $X^*_{\mathcal{D}}$ satisfies linearity and independence of irrelevant comparisons on $X^*_{\mathcal{D}}$.*

Note that $f$ does not satisfy betweenness preservation on $X^*_{\mathcal{D}}$. Indeed, the argument establishing the incompatibility of betweenness preservation and rationalizability for unanimous tournaments in the proof of Theorem 2 only uses preferences in the single-peaked domain $\mathcal{D} = \{123, 213, 231, 321\}$.

The proof in the Appendix actually establishes a stronger statement than Theorem 5. For any $P \in \mathcal{P}$ and $B \in \mathcal{S}_A$, let $P_B$ denote the restriction of $P$ to $B$. For any $\mathcal{D} \subseteq \mathcal{P}$, define $\mathcal{D}_B := \{P_B | P \in \mathcal{D}\}$ and let $X^*_{\mathcal{D}_B}$ denote the set of fractional tournaments on $B$ that are generated by a probability distribution on $\mathcal{D}_B$.

We show that (i) for any domain $\mathcal{D}$ ensuring the existence of a unique admissible extension rule on $X^*_{\mathcal{D}}$, this extension rule satisfies linearity on $X^*_{\mathcal{D}}$, and (ii) for any domain $\mathcal{D}$ ensuring that for every $B \in \mathcal{S}_A$ there is a unique admissible extension rule on $X^*_{\mathcal{D}_B}$, the extension rule on $X^*_{\mathcal{D}}$ satisfies independence of irrelevant comparisons on $X^*_{\mathcal{D}}$.

As an illustration of Theorem 5, suppose that $\mathcal{D}$ is the set of single-peaked preferences with respect to the natural ordering of the alternatives. It is not difficult to check that the unique admissible extension rule $f$ on $X^*_{\mathcal{D}}$ is then given by

$$f_{aB}(x) = x_{a \min\{b \in B | b > a\}} - x_{\max\{b \in B | a > b\} a}$$

for all $x \in X^*_{\mathcal{D}}$, $B \in \mathcal{S}_A$, and $a \in B$ (with the convention $x_{a \min \emptyset} = 1$ and $x_{\max \emptyset a} = 0$). This rule obviously satisfies the two axioms in Theorem 5.

The sequentially binary domains are *maximal* domains generating fractional tournaments with a unique admissible extension. To state this formally, let $\subset$ denote strict inclusion. The proof of the following result is in the Appendix.

Proposition. *If $\mathcal{D}$ is a sequentially binary domain and $\mathcal{D} \subset \mathcal{D}' \subseteq \mathcal{P}$, there exist $\alpha, \alpha' \in \Delta(\mathcal{D}')$ such that $x^*(\alpha) = x^*(\alpha')$ and $y^*(\alpha) \neq y^*(\alpha')$.*

## 6. Further connections to the literature and concluding remarks

Throughout this paper, we interpreted an extension rule as a mechanism for making randomized social choices based on the fractional tournament generated by the voters' preferences. But an *individual choice* interpretation also makes sense: a fractional tournament encodes the randomized binary choices of a single stochastically rational individual, and an extension rule is a procedure for inferring from that information the individual's randomized choices from larger agendas.

It is worth recalling that *deterministic* rationalizable choice functions are completely determined by their restriction to binary agendas. This property is arguably their greatest advantage: it tremendously simplifies both the decision maker's problem and the external analyst's task of predicting the decision maker's choices.[16] Since *stochastically* rationalizable random choice functions too are based on orderings, i.e., on *binary* comparisons, it is natural to inquire to what extent stochastic choices from arbitrary agendas can be inferred from stochastic binary choices.

The corollary to Theorem 4 may be reinterpreted as a partial answer to that question. Under the individual choice interpretation, however, conditions for a unique extension to a rationalizable fractional tournament $x$ need not take the form of restrictions on the support of the distribution of preferences generating $x$. The more general problem, phrased in the language of individual choice theory, consists of determining which submodels of the random utility model possess the "unique extension property" that a random choice function is fully determined by its behavior on the binary agendas.

The issue received some attention in the literature. Marley (1982) notes that in the strict utility model proposed by Luce (1959), the choice probabilities on a subset $B$ of alternatives are a rational function of the binary choice probabilities between alternatives in $B$. He shows that this property is shared by the so-called independent Thurtonian models, and also by some non-independent random utility models. Apesteguia, Ballester, and Lu (2017) show that the unique extension property holds for the single-crossing model.

A related issue is that of identification. The question here is whether the probability distribution generating a random choice function is unique. If $m \geq 4$, a stochastically rationalizable choice function is typically unidentified: see Barberà and Pattanaik (1986), Fishburn (1988), and McClellon (2015). Turansick (2022) offers a complete characterization of the random choice functions that are generated by a unique distribution.

The unique extension issue is linked to that of identification. Indeed, if a rationalizable *fractional tournament* $x$ is identified, i.e., if the probability distribution generating $x$ is unique, then $x$ has a unique stochastically rationalizable extension $y$.

But it is neither necessary nor sufficient that a *random choice function* $y$ be identified to guarantee that it is the unique stochastically rational extension of the fractional tournament $x$ it generates.

To see that identification of $y$ is not sufficient, observe that the three-alternative random choice function $y$ uniquely generated by the distribution $\alpha(123) = \alpha(321) = \frac{1}{2}$ generates the fractional tournament $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which admits the random choice function $y'$ uniquely generated by $\alpha'(132) = \alpha'(231) = \frac{1}{2}$ as an alternative stochastically rational extension. Note that $x$ is not identified, although $y$, $y'$ are.

To see that identification of $y$ is not necessary, consider the four-alternative example discussed in Fishburn (1988). Although the two distributions $\alpha(1234) = \alpha(2143) = \frac{1}{2}$ and

[16] As Moulin (1988, page 306), points out, a binary relation on $A$ is determined by only $m(m-1)/2$ pairwise comparisons whereas a choice function involves nearly $2^m$ free parameters.

$\alpha'(1243) = \alpha(2134) = \frac{1}{2}$ generate the same random choice function $y$, it is easy to see that $y$ is the unique stochastically rationalizable extension of the fractional tournament $x$ it extends. Neither $x$ nor $y$ is identified.

Ultimately, the quest for identified models proceeds from an intention to *predict* choice behavior. As Turansick (2022) argues,

> "Identification guarantees that counterfactual analysis will be accurate up to the choice of model. When choice behavior has multiple representations, counterfactuals may take on different values for each one of these representations."

But there seems to be little room for counterfactuals if the random choice function is completely known. Identification matters when the analyst observes choice frequencies from a *restricted* set of agendas: counterfactual analysis then consists of predicting choice from *other* agendas.[17] An extension rule performs precisely that task for the particular case where choice frequencies are observed for binary agendas. The general problem of extending a random choice function defined on an arbitrary incomplete collection of agendas deserves further study.

## Appendix

### A.1 *More on neutrality*

We check that the leximax extension rule $f^L$ differs from the extension rule $\overline{f}$. Consider the three-alternative rationalizable fractional tournament $x = (x_{12}, x_{23}, x_{31}) = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$. Write any distribution $\alpha \in \Delta(\mathcal{P})$ as $\alpha = (\alpha(123), \alpha(132), \alpha(312), \alpha(321), \alpha(231), \alpha(213))$.

To compute $\overline{f}(x)$, check first that the carriers of $x$ are

$$\mathcal{D}_1 = \{123, 312, 321\},$$
$$\mathcal{D}_2 = \{132, 312, 231\},$$
$$\mathcal{D}_3 = \{312, 213\},$$

and the corresponding decompositions of $x$ are $\alpha^{\mathcal{D}_1} = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, 0, 0)$, $\alpha^{\mathcal{D}_2} = (0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0)$, and $\alpha^{\mathcal{D}_3} = (0, 0, \frac{2}{3}, 0, 0, \frac{1}{3})$. The probability distribution on $\{1, 2, 3\}$ prescribed by $\overline{f}(x)$ is

$$\begin{aligned}
\overline{f}_{\{1,2,3\}}(x) &= \frac{1}{3} f^{\mathcal{D}_1}_{\{1,2,3\}}(x) + \frac{1}{3} f^{\mathcal{D}_2}_{\{1,2,3\}}(x) + \frac{1}{3} f^{\mathcal{D}_3}_{\{1,2,3\}}(x) \\
&= \frac{1}{3}\left(\frac{1}{3}, 0, \frac{2}{3}\right) + \frac{1}{3}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + \frac{1}{3}\left(0, \frac{1}{3}, \frac{2}{3}\right) \\
&= \left(\frac{2}{9}, \frac{2}{9}, \frac{5}{9}\right).
\end{aligned}$$

[17] As Turansick (2022) points out, identification is also important from a theoretical viewpoint:

> "One of the main goals of choice theory is to provide simplified approximations of reality in an attempt to explain observed choice behavior. Identification of a model allows us to do exactly this."

This theoretical motivation for identification is compelling even for choice functions defined on all agendas.

To compute $f^L(x)$, check that minimizing the leximax ordering over $\Delta_x = co\{\alpha^{\mathcal{D}_1}, \alpha^{\mathcal{D}_2}, \alpha^{\mathcal{D}_3}\}$ gives $\alpha_x^L = (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0)$. The probability distribution on $\{1, 2, 3\}$ prescribed by $f^L(x)$ is therefore

$$\begin{aligned} f^L_{\{1,2,3\}}(x) &= \sum_{P \in \mathcal{P}} \alpha_x^L(P) y^*_{\{1,2,3\}}(P) \\ &= \frac{1}{6}(1,0,0) + \frac{1}{6}(1,0,0) + \frac{1}{3}(0,0,1) + \frac{1}{6}(0,0,1) + \frac{1}{6}(0,1,0) \\ &= \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right). \end{aligned}$$

To understand the difference with the distribution prescribed by the rule $\overline{f}$, notice that the latter can be written $\overline{f}_{\{1,2,3\}}(x) = \sum_{P \in \mathcal{P}} \overline{\alpha}_x(P) y^*_{\{1,2,3\}}(P)$ where $\overline{\alpha}_x = (\frac{1}{9}, \frac{1}{9}, \frac{4}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$ maximizes the leximin ordering on $\Delta_x$.

### A.2 *Proof of Theorem 4*

For any $P \in \mathcal{P}$ and $B \in \mathcal{S}_A$, recall that $P_B$ denotes the restriction of $P$ to $B$ and, for any $\mathcal{D} \subseteq \mathcal{P}$, $\mathcal{D}_B = \{P_B | P \in \mathcal{D}\}$. As before, $\subset$ denotes strict inclusion.

Lemma 1. *If $\mathcal{D} \subseteq \mathcal{P}$ is a sequentially binary domain on $A$ and $B \in \mathcal{S}_A$, then $\mathcal{D}_B$ is a sequentially binary domain on $B$.*

Proof. Let $\mathcal{D} \subseteq \mathcal{P}$ be a sequentially binary domain on $A$, $|A| = m$. By definition, there exists a function $g \in G_A$ such that $\mathcal{D}_g = \mathcal{D}$. We fix an alternative $a \in A$, without loss of generality $a = 1$, and prove that there exists a function $g' \in G_{A\setminus\{1\}}$ such that $\mathcal{D}_{A\setminus\{1\}} = \mathcal{D}_{g'}$. The function $g'$ is constructed from $g$ through a sequential process. We define a finite sequence $(V_t, h_t)_{t=1}^T$ such that $S^{(m-1)} = V_1 \supset \cdots \supset V_T = S^{(m-2)}$ and each $h_t$ is a function from $V_t$ to $A$. The function $h_1$ coincides with $g$, and each function $h_{t+1}$ is defined by altering its predecessor $h_t$. The construction ensures that $g' := h_T$ belongs to $G_{A\setminus\{1\}}$ and $\mathcal{D}_{g'} = \mathcal{D}_{A\setminus\{1\}}$.

*Step 1 Preliminaries.*

Call a set $V \subseteq S^{(m-1)}$ *comprehensive* if (i) $[s \in V, s' \in S, s' \prec s] \Rightarrow [s' \in V]$ and (ii) for all $s \in S_0$, $[(s,0) \in V] \Leftrightarrow [(s,1) \in V]$. Note that $S^{(m-1)}$ and $S^{(m-2)}$ are comprehensive. Given a comprehensive set $V$, write $V_0 = V \cup \{s_0\}$ and let $\partial V$ be the set of terminal sequences in $V$, namely $\partial V = \{s \in V | (s,0), (s,1) \notin V\}$. For any function $h : V \to A$ and any $s \in V_0 \setminus \partial V$, let $O_h(s) = \{h(s,0), h(s,1)\}$.

Let $H_V$ be the set of functions $h : V \to A$ satisfying the following properties:

$$h(s) \neq h(s') \quad \text{for all } s, s' \in V \text{ such that } s \prec s', \tag{17}$$

$$h(s,0) \neq h(s,1) \quad \text{for all } s \in V_0 \setminus \partial V, \tag{18}$$

and

$$h(s) \in O_h(tw(s)) \quad \text{for all } s \in V \text{ such that } h(s) \neq 1, \tag{19}$$

$$O_h(s) = O_h(s') \quad \text{for all } s, s' \in V_0 \setminus \partial V \text{ such that} \\ h(WP(s)) \setminus \{1\} = h(WP(s')) \setminus \{1\}. \tag{20}$$

These properties generalize the conditions defining a consistent selection function, namely (13), (14), (15), (16). In particular, observe that $H_{S^{(m-1)}} = G_A$ and $H_{S^{(m-2)}} = G_{A\setminus\{1\}}$.

*Step 2 Defining the sequence* $(V_t, h_t)_{t=1}^T$ *and the function* $g'$.

First, define

$$(V_1, h_1) = (S^{(m-1)}, g).$$

To complete the definition of the sequence, proceed inductively. Let $T := |g^{-1}(1)|$, fix $t \in \{1, \dots, T-1\}$, and suppose $(V_1, h_1), \dots, (V_t, h_t)$ have been defined. To define $(V_{t+1}, h_{t+1})$, we introduce additional notation. For any $s = (s_1, \dots, s_k) \in V_t$, let $NF_t(s) = V_t \setminus \{s' \in V_t | s \prec s'\}$ denote the set of sequences in $V_t$ that do *not follow* $s$. For any $s' = (s'_1, \dots, s'_{k'}) \in S$, let $ss' := (s_1, \dots, s_k, s'_1, \dots, s'_{k'})$ be the sequence obtained by appending $s'$ to $s$. If $S' \subseteq S$, we write $sS' = \{ss' | s' \in S'\}$ (and assume, by convention, $s\emptyset = \emptyset$). The set of *continuations* of $s$ in $V_t$ is $Co_t(s) = \{s' \in S | ss' \in V_t\}$.

Pick a *sequence of last occurrence of* 1 in $V_t$, i.e., a sequence $s^1 \in V_t$ such that $h_t(s^1) = 1$ and $h_t(s) \neq 1$ for all $s \in V_t$ such that $tw(s^1) \prec s$. Define the *direct predecessor* of a node $s = (s_1, \dots, s_k) \in S$ to be $dp(s) = (s_1, \dots, s_{k-1})$ if $k > 1$ and $dp(s) = s_0$ if $k = 1$. Let

$$V_{t+1} = NF_t(dp(s^1)) \cup dp(s^1)Co_t(s^1). \tag{21}$$

Note that the two sets on the right side of this equation are disjoint. For each $s \in dp(s^1)Co_t(s^1)$, let $\sigma(s)$ be the sequence in $Co_t(s^1)$ such that $s = dp(s^1)\sigma(s)$. Define $h_{t+1} : V_{t+1} \to A$ by

$$h_{t+1}(s) = \begin{cases} h_t(s) & \text{if } s \in NF_t(dp(s^1)), \\ h_t(s^1\sigma(s)) & \text{if } s \in dp(s^1)Co_t(s^1). \end{cases} \tag{22}$$

The construction of $(V_{t+1}, h_{t+1})$ is illustrated in Figure 4.

In essence, it consists in picking a sequence $s^1$ of last occurrence of 1 in $V_t$ and replacing the subtree rooted at the direct predecessor of $s^1$ (and the alternatives selected by $h_t$ at the nodes of that subtree) by the subtree rooted at $s^1$ (and the alternatives selected by $h_t$ at the nodes of that subtree).

Observe that $S^{(m-1)} = V_1 \supset \cdots \supset V_T = S^{(m-2)}$, each $V_t$ is comprehensive, and each $h_t$ is a function from $V_t$ to $A$. Moreover, $|h_{t+1}^{-1}(1)| = |h_t^{-1}(1)| - 1$ for each $t \in \{1, \dots, T-1\}$, i.e., the number of nodes where alternative 1 is selected decreases by one at each step along the sequence $(V_t, h_t)_{t=1}^T$. Since $T = |g^{-1}(1)|$, it follows that $|h_T^{-1}(1)| = 0$, meaning that the range of $h_T$ is $A \setminus \{1\}$.

Define $g' : S^{(m-2)} \to A \setminus \{1\}$ by $g' = h_T$.

*Step 3 Proving that* $g' \in G_{A\setminus\{1\}}$.

By definition, $h_1 = g \in G_A = H_{V_1}$. Proceeding inductively, we now fix $t \in \{1, \dots, T-1\}$, assume that $h_t \in H_{V_t}$, and show that $h_{t+1} \in H_{V_{t+1}}$. It then follows that $g' := h_T \in H_{V_T} = G_{A\setminus\{1\}}$.

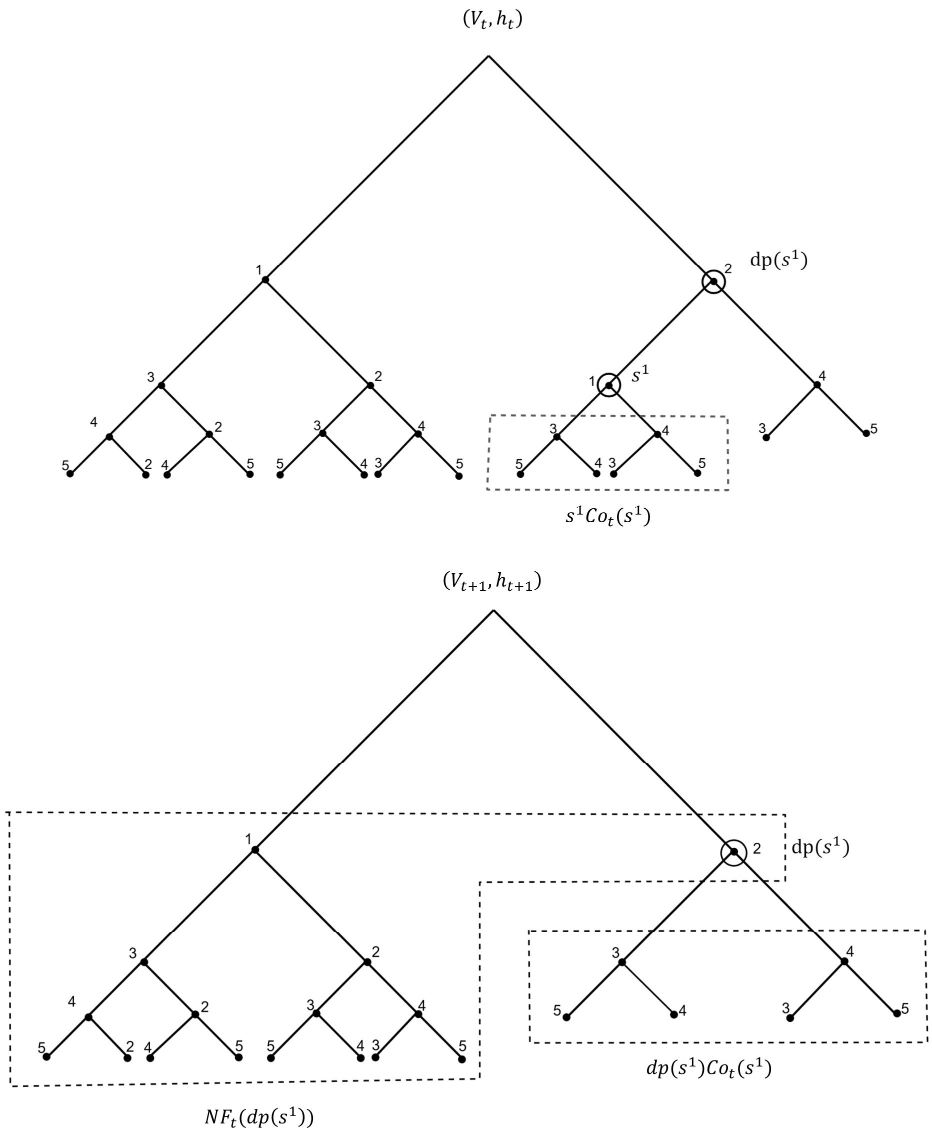


Figure 4. Constructing the sequence $(V_t, h_t)$

Since $h_t : V_t \to A$ satisfies (17) and (18), it is clear that $h_{t+1} : V_{t+1} \to A$ also does. It remains to show that $h_{t+1}$ satisfies (19) and (20).

*Step 3.1 $h_{t+1}$ satisfies* (19).

Let $s \in V_{t+1}$, and suppose $h_{t+1}(s) \neq 1$ and $tw(s) \in V_{t+1} \setminus \partial V_{t+1}$. We claim that $h_{t+1}(s) \in O_{h_{t+1}}(tw(s))$.

If $s \in NF_t(dp(s^1))$, then $h_{t+1}(s) = h_t(s)$. Since $tw(s) \notin \partial V_{t+1}$, we have $tw(s) \notin \partial V_t$. By the induction hypothesis, $h_t(s) \in O_{h_t}(s)$. Since $O_{h_t}(tw(s)) = O_{h_{t+1}}(tw(s))$, the claim follows.

If $s \in dp(s^1)Co_t(s^1)$, there exists $\sigma(s)$ such that $s = dp(s^1)\sigma(s)$ and we have $h_{t+1}(s) = h_t(s^1\sigma(s))$. By the induction hypothesis, $h_t(s^1\sigma(s)) \in O_{h_t}(tw(s^1\sigma(s))) = O_{h_{t+1}}(tw(s)))$, and the claim follows again.

*Step 3.2 $h_{t+1}$ satisfies* (20).

Let $s, s' \in V_{t+1} \setminus \partial V_{t+1}$ and suppose

$$h_{t+1}\big(WP(s)\big) \setminus \{1\} = h_{t+1}\big(WP(s')\big) \setminus \{1\}. \tag{23}$$

We claim that $O_{h_{t+1}}(s) = O_{h_{t+1}}(s')$.

*Case (i)* $s, s' \in NF_t(dp(s^1))$.

Then $v \in NF_t(dp(s^1))$ for all $v \in WP(s) \cup WP(s')$. From (22), $h_{t+1}(WP(s)) = h_t(WP(s))$ and $h_{t+1}(WP(s')) = h_t(WP(s'))$. Hence, by (23), $h_t(WP(s)) \setminus \{1\} = h_t(WP(s')) \setminus \{1\}$. By the induction hypothesis, $O_{h_t}(s) = O_{h_t}(s')$. Since by definition $O_{h_{t+1}}(s) = O_{h_t}(s)$ and $O_{h_{t+1}}(s') = O_{h_t}(s')$, the claim follows.

*Case (ii)* $s, s' \in dp(s^1)Co_t(s^1)$.

Then there exist $\sigma(s), \sigma(s') \in Co_t(s^1)$ such that $s = dp(s^1)\sigma(s)$ and $s' = dp(s^1)\sigma(s')$. For any two sets $A, B$, write $C = A \uplus B$ if $C = A \cup B$ and $A \cap B = \emptyset$. Observe that $WP(s) = WP(dp(s^1)) \uplus dp(s^1)WP(\sigma(s))$. Since $WP(dp(s^1)) \subseteq NF_t(dp(s^1))$, (22) implies $h_{t+1}(WP(dp(s^1))) = h_t(WP(dp(s^1)))$. Since $dp(s^1)WP(\sigma(s)) \subseteq dp(s^1)Co_t(s^1)$, (22) implies $h_{t+1}(dp(s^1)WP(\sigma(s))) = h_t(s^1WP(\sigma(s)))$. Hence,

$$h_{t+1}\big(WP(s)\big) = h_t\big(WP\big(dp\big(s^1\big)\big)\big) \uplus h_t\big(s^1WP\big(\sigma(s)\big)\big).$$

Likewise,

$$h_{t+1}\big(WP\big(s'\big)\big) = h_t\big(WP\big(dp\big(s^1\big)\big)\big) \uplus h_t\big(s^1WP\big(\sigma\big(s'\big)\big)\big).$$

Combining these statements with (23), we get $h_t(s^1WP(\sigma(s))) \setminus \{1\} = h_t(s^1WP(\sigma(s'))) \setminus \{1\}$. It follows that

$$\big[h_t\big(WP\big(s^1\big)\big) \uplus h_t\big(s^1WP\big(\sigma(s)\big)\big)\big] \setminus \{1\} = \big[h_t\big(WP\big(s^1\big)\big) \uplus h_t\big(s^1WP\big(\sigma\big(s'\big)\big)\big)\big] \setminus \{1\};$$

hence, $[h_t(WP(s^1\sigma(s)))] \setminus \{1\} = [h_t(WP(s^1\sigma(s')))] \setminus \{1\}$. By the induction hypothesis, $O_{h_t}(s^1\sigma(s)) = O_{h_t}(s^1\sigma(s'))$, and the claim follows by definition of $h_{t+1}$.

*Case (iii)* $s \in NF_t(dp(s^1))$ and $s' \in dp(s^1)Co_t(s^1)$.

In this case, we have

$$h_{t+1}\big(WP(s)\big) = h_t\big(WP(s)\big),$$
$$h_{t+1}\big(WP\big(s'\big)\big) = h_t\big(WP\big(dp\big(s^1\big)\big)\big) \uplus h_t\big(s^1WP\big(\sigma\big(s'\big)\big)\big),$$

and (23) implies

$$\begin{aligned} h_t\big(WP(s)\big) \setminus \{1\} &= \big[h_t\big(WP\big(dp\big(s^1\big)\big)\big) \setminus \{1\}\big] \uplus \big[h_t\big(s^1WP\big(\sigma\big(s'\big)\big)\big) \setminus \{1\}\big] \\ &= \big[h_t\big(WP\big(s^1\big)\big) \setminus \{1\}\big] \uplus \big[h_t\big(s^1WP\big(\sigma\big(s'\big)\big)\big) \setminus \{1\}\big] \\ &= h_t\big(WP\big(s^1\sigma\big(s'\big)\big)\big) \setminus \{1\}, \end{aligned}$$

where the second equality holds because $h_t(s^1) = 1$. By the induction hypothesis, $O_{h_t}(s) = O_{h_t}(s^1\sigma(s'))$, and the claim follows again by definition of $h_{t+1}$.

*Step 4 Proving that* $\mathcal{D}_{g'} = \mathcal{D}_{A\setminus\{1\}}$.

For each $t \in \{1, \dots, T\}$, every sequence $s \in \partial V_t$ has length $m-1$ or $m-2$. If $s$ has length $m-1$, say, $s = (s_1, \dots, s_{m-1})$, then

$$P_{h_t}(s) := a_{h_t}(s)h_t(s_1, \dots, s_{m-1}) \cdots h_t(s_1, s_2)h_t(s_1) \in \mathcal{P}_A = \mathcal{P},$$

where $a_{h_t}(s)$ is the unique alternative in $A \setminus \{h_t(s_1, \dots, s_{m-1}), \dots, h_t(s_1, s_2), h_t(s_1)\}$. If $s$ has length $m-2$, say, $s = (s_1, \dots, s_{m-2})$, then $h_t(WP(s)) = A \setminus \{1\}$ and

$$P_{h_t}(s) := a_t(s)h_t(s_1, \dots, s_{m-2}) \cdots h_t(s_1, s_2)h_t(s_1) \in \mathcal{P}_{A\setminus\{1\}},$$

where $a_{h_t}(s)$ is the unique alternative in $A \setminus \{h_t(s_1, \dots, s_{m-2}), \dots, h_t(s_1, s_2), h_t(s_1)\}$.

Let $P_{t,1}(s)$ be the restriction of $P_{h_t}(s)$ to $A \setminus \{1\}$ and define

$$\mathcal{D}_{t,1} = \{P_{t,1}(s) | s \in \partial V_t\}.$$

Note that $\mathcal{D}_{1,1} = \mathcal{D}_g = \mathcal{D}_{A\setminus\{1\}}$ and $\mathcal{D}_{T,1} = \mathcal{D}_{g'}$. To prove that $\mathcal{D}_{g'} = \mathcal{D}_{A\setminus\{1\}}$, it therefore suffices to establish that $\mathcal{D}_{t+1,1} = \mathcal{D}_{t,1}$ for each $t = 1, \dots, T-1$.

Fix $t \in \{1, \dots, T-1\}$ and recall the definition of $V_{t+1}$ from (21), where $s^1$ is a sequence of last occurrence of 1 in $V_t$. Let $WF_t(s^1) = \{s \in V_t | s^1 \precsim s\}$ and $WF_{t+1}(dp(s^1)) = \{s \in V_{t+1} | dp(s^1) \precsim s\}$. Partition $\partial V_t$ into the following components:

$$\begin{aligned} A_t &= \partial V_t \cap NF_t(dp(s^1)), \\ B_t &= \partial V_t \cap WF_t(s^1), \\ C_t &= \partial V_t \cap NF_t(tw(s^1)), \end{aligned}$$

and partition $\partial V_{t+1}$ into the components:

$$\begin{aligned} A_{t+1} &= \partial V_{t+1} \cap NF_{t+1}(dp(s^1)), \\ \widetilde{B}_{t+1} &= \partial V_{t+1} \cap WF_{t+1}(dp(s^1)). \end{aligned}$$

Note that some of the components may be empty.

Recalling the definition of $h_{t+1}$ given in (22), we make three observations.

First, $A_{t+1} = A_t$ and

$$P_{t,1}(s) = P_{t+1,1}(s) \quad \text{for all } s \in A_{t+1} = A_t. \tag{24}$$

Second, let $\partial Co_t(s^1) = \{s' \in S | s^1 s' \in \partial V_t\}$. By definition, $B_t = s^1 \partial Co_t(s^1)$ and $\widetilde{B}_{t+1} = dp(s^1)\partial Co_t(s^1)$. For each $\sigma \in \partial Co_t(s^1)$, we have $s^1\sigma \in B_t$, $dp(s^1)\sigma \in \widetilde{B}_{t+1}$, and $P_{t,1}(s^1\sigma) = P_{t+1}(dp(s^1)\sigma)$. It follows that

$$\{P_{t,1}(s) | s \in B_t\} = \{P_{t+1,1}(s) | s \in \widetilde{B}_{t+1}\}. \tag{25}$$

Third, the orderings $P_{t,1}(s)$ associated with the sequences $s \in C_t$ are redundant. Indeed, because $h_t$ satisfies (19), there exists $\widetilde{s} \in \{(s^1, 0), (s^1, 1)\}$ such that $h_t(\widetilde{s}) = h_t(tw(s^1))$. By (20), $O_{h_t}(\widetilde{s}) = O_{h_t}(tw(s^1))$, and it follows that

$$\begin{aligned}\{P_{t,1}(s)|s \in C_t\} &= \{P_{t,1}(s)|tw(s^1) \prec s\} \\ &= \{P_{t,1}(s)|\widetilde{s} \prec s\} \\ &\subseteq \{P_{t,1}(s)|s \in B_t\}.\end{aligned} \tag{26}$$

It follows from (24), (25), (26) that $\mathcal{D}_{t,1} = \{P_{t,1}(s)|s \in \partial V_t = A_t \cup B_t \cup C_t\} = \{P_{t+1,1}(s)| s \in \partial V_{t+1} = A_{t+1} \cup \widetilde{B}_{t+1}\} = \mathcal{D}_{t+1,1}$. □

Proof of Theorem 4. Theorem 4 is trivially true if $m := |A| = 2$. Proceeding by induction, fix $\overline{m} > 2$ and make the induction hypothesis that Theorem 4 is true whenever $m < \overline{m}$. Fix $A$ such that $|A| = \overline{m}$, say, $A = \{1, \dots, \overline{m}\}$. Let $\mathcal{D} \subseteq \mathcal{P}$ be a sequentially binary domain on $A$, and let $\alpha, \alpha' \in \Delta(\mathcal{D})$ be such that $x^*(\alpha) = x^*(\alpha')$.

*Step 1* $y^*_B(\alpha) = y^*_B(\alpha')$ *for all* $B \subset A$.

Let $B \subset A$ and define $\alpha_B, \alpha'_B \in \Delta(\mathcal{D}_B)$ as follows: for all $\widetilde{P} \in \mathcal{D}_B$,

$$\alpha_B(\widetilde{P}) = \sum_{P \in \mathcal{D}: P_B = \widetilde{P}} \alpha(P) \quad \text{and} \quad \alpha'_B(\widetilde{P}) = \sum_{P \in \mathcal{D}: P_B = \widetilde{P}} \alpha'(P). \tag{27}$$

Let $x^*(\alpha_B)$, $x^*(\alpha'_B)$ be the fractional tournaments on $B$ generated by $\alpha_B$, $\alpha'_B$, and let $y^*(\alpha_B)$, $y^*(\alpha'_B)$ be the random choice functions on $B$ generated by $\alpha_B$, $\alpha'_B$. Since $x^*(\alpha) = x^*(\alpha')$, we have $x^*(\alpha_B) = x^*(\alpha'_B)$. Since, by Lemma 1, $\mathcal{D}_B$ is a sequentially binary domain on $B$, the induction hypothesis implies that $y^*(\alpha_B) = y^*(\alpha'_B)$. This in turn implies

$$y^*_B(\alpha) = y^*_B(\alpha').$$

*Step 2* $y^*_A(\alpha) = y^*_A(\alpha')$.

Because $\mathcal{D}$ is a sequentially binary domain on $A$, there exists $g \in G_A$ such that $\mathcal{D} = \mathcal{D}_g$. Without loss of generality, assume that $g((0)) = 1$ and $g((1)) = 2$. For any $P \in \mathcal{P}$ and $B \in \mathcal{S}_A$, let $\max_B P$ denote the best alternative in $B$ according to $P$.

Because of (15), there is a unique ordering $P \in \mathcal{D}_g = \mathcal{D}$ such that $\max_A P = 1$. Call this ordering $P_{(1)}$. Likewise, let $P_{(2)}$ denote the unique ordering $P \in \mathcal{D}$ such that $\max_A P = 2$. Observe that for all $P \in \mathcal{D}$,

$$\max_{A \setminus \{2\}} P = 1 \quad \Leftrightarrow \quad P = P_{(1)}, \tag{28}$$

$$\max_{A \setminus \{1\}} P = 2 \quad \Leftrightarrow \quad P = P_{(2)}. \tag{29}$$

From (28), we have

$$y^*_{1A}(\alpha) = y^*_{1A \setminus \{2\}}(\alpha) = \alpha(P_{(1)}), \tag{30}$$

$$y^*_{1A}(\alpha') = y^*_{1A \setminus \{2\}}(\alpha') = \alpha'(P_{(1)}). \tag{31}$$

Since $y^*_{1A\setminus\{2\}}(\alpha) = y^*_{1A\setminus\{2\}}(\alpha')$ by Step 1, we conclude that $y^*_{1A}(\alpha) = y^*_{1A}(\alpha')$. Likewise, it follows from (29) that $y^*_{2A}(\alpha) = y^*_{2A}(\alpha')$.

To complete the proof, consider now any $a \in A \setminus \{1, 2\}$. Distinguish two cases.

*Case (i)* $\max_{A\setminus\{1\}} P_{(1)} \neq a$ or $\max_{A\setminus\{2\}} P_{(2)} \neq a$.

Without loss of generality, suppose $\max_{A\setminus\{1\}} P_{(1)} \neq a$. This means that $a$ is not ranked second in $P_{(1)}$. Since $P_{(1)}$ is the only $P \in \mathcal{D}$ such that $\max_A P = 1$, it follows that for every $P \in \mathcal{D}$,

$$\max_A P = a \quad \Leftrightarrow \quad \max_{A\setminus\{1\}} P = a.$$

Therefore,

$$y^*_{aA}(\alpha) = y^*_{aA\setminus\{1\}}(\alpha),$$
$$y^*_{aA}(\alpha') = y^*_{aA\setminus\{1\}}(\alpha').$$

Since $y^*_{aA\setminus\{1\}}(\alpha) = y^*_{aA\setminus\{1\}}(\alpha')$ by Step 1, we conclude that $y^*_{aA}(\alpha) = y^*_{aA}(\alpha')$.

*Case (ii)* $\max_{A\setminus\{1\}} P_{(1)} = a$ and $\max_{A\setminus\{2\}} P_{(2)} = a$.

Then, for every $P \in \mathcal{D}$, (15) implies

$$\max_{A\setminus\{1\}} P = a \quad \Leftrightarrow \quad \Big[\text{either } \max_A P = a \text{ or } P = P_{(1)}\Big].$$

It follows that

$$y^*_{aA}(\alpha) = y^*_{aA\setminus\{1\}}(\alpha) - \alpha(P_{(1)}),$$
$$y^*_{aA}(\alpha') = y^*_{aA\setminus\{1\}}(\alpha') - \alpha'(P_{(1)}).$$

Using (30), (31), we obtain[18]

$$y^*_{aA}(\alpha) = y^*_{aA\setminus\{1\}}(\alpha) - y^*_{1A\setminus\{2\}}(\alpha),$$
$$y^*_{aA}(\alpha') = y^*_{aA\setminus\{1\}}(\alpha') - y^*_{1A\setminus\{2\}}(\alpha'),$$

and it follows again from Step 1 that $y^*_{aA}(\alpha) = y^*_{aA}(\alpha')$. □

### A.3 *Proof of Theorem 5*

Let $\mathcal{D} \subseteq \mathcal{P}$ be a sequentially binary domain and let $f : X^*_{\mathcal{D}} \to Y^*_{\mathcal{D}}$ be the unique admissible extension rule on $X^*_{\mathcal{D}}$.

[18]Permuting 1 and 2 in the above argument leads to the equally valid formulas:

$$y^*_{aA}(\alpha) = y^*_{aA\setminus\{2\}}(\alpha) - y^*_{2A\setminus\{1\}}(\alpha),$$
$$y^*_{aA}(\alpha') = y^*_{aA\setminus\{2\}}(\alpha') - y^*_{2A\setminus\{1\}}(\alpha').$$

*Step 1 $f$ satisfies linearity on $X^*_{\mathcal{D}}$.*

The crucial observation is that the map $y^* : \Delta(\mathcal{D}) \to Y$, $y^*(\alpha) = \sum_{P\in\mathcal{D}} \alpha(P)y^*(P)$, is an affine function. That is,

$$y^*\big(\lambda\alpha + (1-\lambda)\alpha'\big) = \lambda y^*(\alpha) + (1-\lambda)y^*\big(\alpha'\big) \tag{32}$$

for all $\alpha, \alpha' \in \Delta(\mathcal{D})$ and $\lambda \in [0, 1]$.

To prove that $f : X^*_{\mathcal{D}} \to Y$ is an affine function, fix $x, x' \in X^*_{\mathcal{D}}$ and $\lambda \in [0, 1]$. Let $\alpha, \alpha' \in \Delta(\mathcal{D})$ be any probability distributions on $\mathcal{D}$ such that $x = x^*(\alpha)$ and $x' = x^*(\alpha')$. Since $y^*(\alpha)$, $y^*(\alpha')$ are admissible extensions of $x$, $x'$, and $f$ is the unique admissible extension rule on $X^*_{\mathcal{D}}$,

$$f(x) = y^*(\alpha) \quad \text{and} \quad f\big(x'\big) = y^*\big(\alpha'\big). \tag{33}$$

Next, observe that $\lambda x + (1-\lambda)x' = x^*(\lambda\alpha + (1-\lambda)\alpha')$. Since $\lambda\alpha + (1-\lambda)\alpha' \in \Delta(\mathcal{D})$, $y^*(\lambda\alpha + (1-\lambda)\alpha')$ is an admissible extension of $\lambda x + (1-\lambda)x'$ and, since $f$ is the unique admissible extension rule on $X^*_{\mathcal{D}}$,

$$f\big(\lambda x + (1-\lambda)x'\big) = y^*\big(\lambda\alpha + (1-\lambda)\alpha'\big). \tag{34}$$

Combining (32), (33), and (34) yield $f(\lambda x + (1-\lambda)x') = \lambda f(x) + (1-\lambda)f(x')$.

*Step 2 $f$ satisfies independence of irrelevant comparisons on $X^*_{\mathcal{D}}$.*

Let $x, x' \in X^*_{\mathcal{D}}$, $B \in \mathcal{S}_A$, and assume $x_B = x'_B$. Let $\alpha, \alpha' \in \Delta(\mathcal{D})$ be such that $x = x^*(\alpha)$ and $x' = x^*(\alpha')$, so that, in particular,

$$x^*_B(\alpha) = x^*_B\big(\alpha'\big). \tag{35}$$

By definition of $f$,

$$f_B(x) = y^*_B(\alpha) \quad \text{and} \quad f_B\big(x'\big) = y^*_B\big(\alpha'\big). \tag{36}$$

We prove that $y^*_B(\alpha) = y^*_B(\alpha')$.

Let $\alpha_B, \alpha'_B \in \Delta(\mathcal{D}_B)$ be the probability distributions defined in (27) and let $x^*(\alpha_B)$, $x^*(\alpha'_B)$ be the fractional tournaments on $B$ generated by $\alpha_B, \alpha'_B$. Since for all distinct $a, b \in B$,

$$x^*_{ab}(\alpha_B) = \sum_{\widetilde{P}\in\mathcal{D}_B : a\widetilde{P}b} \alpha_B(\widetilde{P}) = \sum_{\widetilde{P}\in\mathcal{D}_B : a\widetilde{P}b} \sum_{P\in\mathcal{D} : P_B = \widetilde{P}} \alpha(P) = \sum_{P\in\mathcal{D} : aPb} \alpha(P) = x^*_{ab}(\alpha),$$

we have $x^*_B(\alpha) = x^*(\alpha_B)$. Likewise, $x^*_B(\alpha') = x^*(\alpha'_B)$. Hence, from (35),

$$x^*(\alpha_B) = x^*\big(\alpha'_B\big). \tag{37}$$

Let $y^*(\alpha_B)$, $y^*(\alpha'_B)$ be the random choice functions on $B$ generated by $\alpha_B, \alpha'_B$. Since, by Lemma 1, $\mathcal{D}_B$ is a sequentially binary domain on $B$, (37) and Theorem 4 imply

$$y^*(\alpha_B) = y^*\big(\alpha'_B\big). \tag{38}$$

Since for all $a \in B$,

$$y^*_{aB}(\alpha_B) = \sum_{\substack{\widetilde{P} \in \mathcal{D}_B: \\ a = \max_B \widetilde{P}}} \alpha_B(\widetilde{P}) = \sum_{\substack{\widetilde{P} \in \mathcal{D}_B: \\ a = \max_B \widetilde{P}}} \sum_{P \in \mathcal{D}: P_B = \widetilde{P}} \alpha(P) = \sum_{\substack{P \in \mathcal{D}: \\ a = \max_B P}} \alpha(P) = y^*_{aB}(\alpha),$$

we have $y^*_B(\alpha) = y^*(\alpha_B)$. Likewise, $y^*_B(\alpha') = y^*(\alpha'_B)$. From (38), we conclude that $y^*_B(\alpha) = y^*_B(\alpha')$, as was to be proved. □

## A.4 *Proof of the Proposition*

The proof relies on two lemmas. The first establishes a richness property of the sequentially binary domains.

Lemma 2. *For any sequentially binary domain $\mathcal{D} \subseteq \mathcal{P}$ and any distinct $a, b \in A$, there exist distinct $P, P' \in \mathcal{D}$ such that $cPd \Leftrightarrow cP'd$ for all $\{c, d\} \neq \{a, b\}$.*

Proof. Let $\mathcal{D}$ be a sequentially binary domain and let $a, b \in A$. Let $g$ be a consistent selection function such that $\mathcal{D} = \mathcal{D}_g$.

*Step 1 There exists $s^{ab} \in S_0^{(m-2)}$ such that $O_g(s^{ab}) = \{a, b\}$.*

For any $c \in A$, define $k(c) := \min\{k \in \{1, \dots, m-1\} | c \in O_g(s) \text{ for some } s \in S^{k-1}\}$. Without loss of generality, assume $k(a) \leq k(b)$. Also without loss of generality, suppose that for all $s \in S_0^{(m-2)}$,

$$a \in O_g(s) \quad \Rightarrow \quad g(s, 0) = a. \tag{39}$$

By definition of $k(a)$, there exists $s^a \in S^{k(a)-1}$ such that $g(s^a, 0) = a$. Because $g$ satisfies (15),

$$g(s^a, 1, 0) = \cdots = g\big(\underbrace{s^a, 1, \dots, 1, 0}_{m-1}\big) = a. \tag{40}$$

Since $k(a) \leq k(b)$, $g(s) \neq b$ for all $s \precsim s^a$. By (40) and because $g$ satisfies (13), there exists $l \in \{1, \dots, m - k(a)\}$ such that

$$g\big(s^a, \underbrace{1, \dots, 1}_{l}\big) = b.$$

Let $s^{ab} = (s^a, \underbrace{1, \dots, 1}_{l-1})$. By (40), $O_g(s^{ab}) = \{g(s^a, \underbrace{1, \dots, 1, 0}_{l}), g(s^a, \underbrace{1, \dots, 1}_{l})\} = \{a, b\}$.

*Step 2 There exist distinct $P, P' \in \mathcal{D}$ such that $cPd \Leftrightarrow cP'd$ for all $\{c, d\} \neq \{a, b\}$.*

Let $s^{ab} \in S_0^{(m-2)}$ be such that $O_g(s^{ab}) = \{a, b\}$, say, $g(s^{ab}, 0) = a$ and $g(s^{ab}, 1) = b$. Since $g$ satisfies (15), $a \in O_g(s^{ab}, 1)$ and $b \in O_g(s^{ab}, 0)$. Without loss, assume

$$g\big(s^{ab}, 1, 0\big) = a, \qquad g\big(s^{ab}, 0, 1\big) = b. \tag{41}$$

Let $k(a, b)$ be the length of the sequences $(s^{ab}, 1, 0)$ and $(s^{ab}, 0, 1)$. For any $P = a_1 a_2 \cdots a_m \in \mathcal{D}$ and $k \in \{2, \dots, m\}$, let $T_k(P) := \{a_{m-k+1}, \dots, a_m\}$ denote the set containing the $k$ lowest-ranked alternatives in $P$, the "$k$-tail" of $P$.

Fix $s, s' \in S^{m-1}$ such that $(s^{ab}, 1, 0) \precsim s$, $(s^{ab}, 0, 1) \precsim s'$, and write $P_g(s) = P$, $P_g(s') = P'$. By definition, $P \neq P'$, $T_{k(a,b)}(P) = T_{k(a,b)}(P') =: T_{k(a,b)}$, and $cPd \Leftrightarrow cP'd$ for all $\{c, d\} \subseteq T_{k(a,b)}$ such that $\{c, d\} \neq \{a, b\}$. If $|k(a, b)| = m$, we are done.

If $|k(a, b)| < m$, a simple induction completes the proof. Since $g(WP(s^{ab}, 1, 0)) = g(WP(s^{ab}, 0, 1))$, condition (16) implies

$$O_g\big(s^{ab}, 0, 1\big) = O_g\big(s^{ab}, 1, 0\big).$$

Pick $c \in O_g(s^{ab}, 0, 1) = O_g(s^{ab}, 1, 0)$. Without loss, assume $g(s^{ab}, 0, 1, 0) = g(s^{ab}, 1, 0, 0) = c$. For any $s, s' \in S^{m-1}$ such that $(s^{ab}, 0, 1) \precsim s$ and $(s^{ab}, 1, 0) \precsim s'$, write $P_g(s) = P$, $P_g(s') = P'$, and observe that $T_{k(a,b)+1}(P) = T_{k(a,b)+1}(P') =: T_{k(a,b)+1}$, and $cPd \Leftrightarrow cP'd$ for all $\{c, d\} \subseteq T_{k(a,b)+1}$ such that $\{c, d\} \neq \{a, b\}$. Repeating this argument eventually produces distinct $P, P' \in \mathcal{D}$ such that $cPd \Leftrightarrow cP'd$ for all $\{c, d\} \neq \{a, b\}$. □

The next lemma requires additional terminology. Recall that for any $P \in \mathcal{P}$ and $B \in \mathcal{S}_A$, $P_B$ denotes the restriction of $P$ to $B$. Call $B$ a *triple* if $|B| = 3$. Following Puppe and Slinko (2022), call $\mathcal{D} \subseteq \mathcal{P}$ an *Arrow single-peaked (ASP) domain* if $\mathcal{D}_B := \{P_B | P \in \mathcal{D}\}$ is a single-peaked domain for every triple $B$.

Lemma 3. *Every sequentially binary domain is a maximal ASP domain.*

Proof. Let $\mathcal{D} \subseteq \mathcal{P}$ be a sequentially binary domain. For any triple $B \in \mathcal{S}_A$, Lemma 1 implies that $\mathcal{D}_B$ is a sequentially binary domain on $B$, hence (as noted in the paragraph following Definition 3), a (maximal) single-peaked domain on $B$. It follows that $\mathcal{D}$ is an ASP domain. Slinko (2019) shows that all maximal ASP domains have cardinality $2^{m-1}$. Since $|\mathcal{D}| = 2^{m-1}$, $\mathcal{D}$ is a maximal ASP domain. □

Proof of the Proposition. Let $\mathcal{D}$ be a sequentially binary domain and let $\mathcal{D} \subset \mathcal{D}' \subseteq \mathcal{P}$. By Lemma 3, $\mathcal{D}$ is a maximal ASP domain. Since $\mathcal{D} \subset \mathcal{D}'$, $\mathcal{D}'$ is not an ASP domain. Thus, there exists a triple $B \subseteq A$ such that $\mathcal{D}_B$ is a (maximal) single-peaked domain on $B$ and $\mathcal{D}_B \subset \mathcal{D}'_B$. Choose $Q^0 \in \mathcal{D}' \setminus \mathcal{D}$ such that $Q^0_B \in \mathcal{D}'_B \setminus \mathcal{D}_B$. Without loss of generality, suppose that $B = \{1, 2, 3\}$ and

$$\mathcal{D}_{\{1,2,3\}} = \{123, 213, 231, 321\}, \qquad Q^0_{\{1,2,3\}} = 132.$$

Let $P^0 \in \mathcal{D}$ be such that $P^0_{\{1,2,3\}} = 123$. Since $P^0 \neq Q^0$, there exist $K \geq 1$ pairs of alternatives $\{a^1, b^1\}, \dots, \{a^K, b^K\}$ such that (i) $a^k Q^0 b^k P^0 a^k$ if $k \in \{1, \dots, K\}$, and (ii) $aP^0 b \Leftrightarrow aQ^0 b$ if $\{a, b\} \neq \{a^k, b^k\}$ for all $k \in \{1, \dots, K\}$.

For each $k \in \{1, \dots, K\}$, Lemma 2 ensures that there exist $P^k, Q^k \in \mathcal{D}$ such that $a^k P^k b^k Q^k a^k$ and $aP^k b \Leftrightarrow aQ^k b$ if $\{a, b\} \neq \{a^k, b^k\}$. Define the probability distributions $\alpha, \alpha' \in \Delta(\mathcal{D}')$ by

$$\alpha(P) = \frac{1}{K+1}\big|\big\{k \in \{0, 1, \dots, K\} | P^k = P\big\}\big|,$$

$$\alpha'(P) = \frac{1}{K+1}\big|\big\{k \in \{0, 1, \dots, K\} | Q^k = P\big\}\big|$$

for all $P \in \mathcal{D}'$.

It is straightforward to check that $x^*(\alpha) = x^*(\alpha')$. To complete the proof, we show that $y^*(\alpha) \neq y^*(\alpha')$. Since $P^0_{\{1,2,3\}} = 123$ and $Q^0_{\{1,2,3\}} = 132$, (i) there exists $k \in \{1, \dots, K\}$ such that $\{a^k, b^k\} = \{2, 3\}$ and (ii) $\{a^k, b^k\} \neq \{1, 2\}, \{1, 3\}$ for all $k \in \{1, \dots, K\}$. Without loss of generality, assume $\{a^1, b^1\} = \{2, 3\}$. Note that $P^k_{\{1,2,3\}} = Q^k_{\{1,2,3\}}$ for all $k \in \{2, \dots, K\}$. Thus,

$$\max_{\{1,2,3\}} P^0 = 1 = \max_{\{1,2,3\}} Q^0,$$

$$\max_{\{1,2,3\}} P^1 = 2 \neq 3 = \max_{\{1,2,3\}} Q^1,$$

$$\max_{\{1,2,3\}} P^k = \max_{\{1,2,3\}} Q^k \quad \text{for all } k \in \{2, \dots, K\}.$$

It follows that $y^*_{2\{1,2,3\}}(\alpha) < y^*_{2\{1,2,3\}}(\alpha')$ and $y^*_{3\{1,2,3\}}(\alpha) > y^*_{3\{1,2,3\}}(\alpha')$. □

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